CHAPTER 2

COVARIANCE OF NOISE IN MULTIPLE BEAMS OF ORT

As discussed in chapter 1, the voltage outputs of the 22 modules of the ORT have been combined with different phase gradients to form 12 simultaneous beams both for the total power system and for the correlator system. As the ORT tracks the moon during occultation observations, none of the beams may be pointing exactly towards the position of a source being observed, and in that case the source response may appear within two or more neighbouring beams (see fig. 2.1). Now a question arises in such a case: can we improve the final signal to noise ratio by combining the signal present in both such beams? Before this question can be answered we should realize that even the temporal noise in these two neighbouring beams may not be statistically independent, because the contribution of noise from different modules is the same, only it is combined with a different phase distribution. In fact the question is much more general, namely what is the correlation co-efficient of noise among two simultaneous beams, looking towards different directions with identical power patterns, for a phased array system?

We shall analyse the problem for a case of one-dimensional array; the result can be easily generalized to the case of a 2-dimensional phased array system (Singal, 1985). We assume a north-south array with N equally spaced elements. The voltage signal from all the N elements can be combined and passed through a square law detector to form a total power system. Alternatively the array can be divided into two parts with n elements on the north and the remaining m=N-n elements on the south. Then the voltage signals from the n north elements are added together and similarly signals from the m south elements are added together. These separately combined signals from the north and from the south arms can be

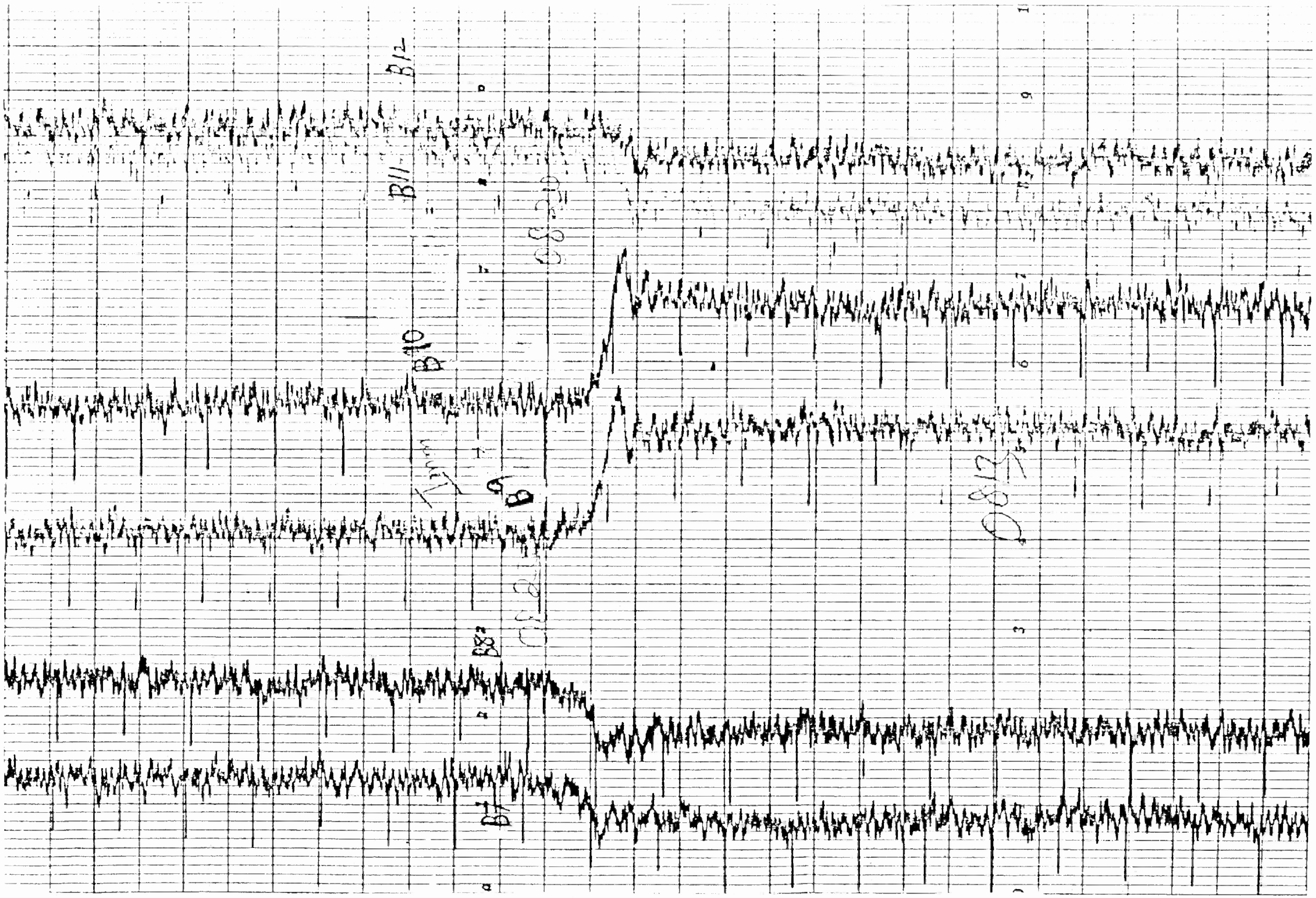

Fig 2.1 Occultation observations of a source which appeared simultaneously in more than one beam of ORT

now multiplied together to form a correlator system. Multiple beams can be formed in either case by using appropriate phase gradients along the array.

We assume a narrow-band system of a predetection band-width $\Delta\nu$ Hz, so that $\Delta\nu/\nu_c \ll 1$, where ν_c is the centre frequency of the pass-band. For the ORT system $\Delta\nu$ = 4MHz and ν_c = 326.5 MHz. We also make the reasonable assumption that any further addition to noise in a beam after the beam formation network is negligible.

For our purpose we shall consider two beams, one being the central beam formed with no phase-gradients across the array and the other called the neighbouring beam formed by introducing a successive phase difference, $\Delta\phi$, between two adjacent elements, given by

$$\Delta\phi = \frac{d\,\omega_c}{c} \sin\delta$$

Where δ is the angle of pointing of the neighbouring beam with respect to that of the central beam, d is the distance between two adjacent elements, c is the velocity of electromagnetic waves in the medium (vacuum!) and $\omega_c = 2\pi\nu_c$. The instrumental phase delay as calculated above, is introduced to compensate for the geometric phase delay between adjacent elements for a signal coming from direction δ.

2.1 NOISE REPRESENTATION

Narrow-band noise voltage output of an individual element can be represented as $V(t)\cos(\omega_c t+\phi(t))$, where the variations of envelope $V(t)$ and the phase $\phi(t)$ are slow compared to those of $\cos(\omega_c t)$ and are on a time scale $\sim 1/\Delta\nu$. The probability density of $V(t)$ is Rayleigh distributed, and $\phi(t)$ is uniformly distributed inside the range 0 to 2π. Moreover the variance of the noise power $V^2(t)/2$, which has a non-zero mean, is equal to the square of the mean value of $V^2(t)/2$, i.e.,

Variance of $V^2(t)/2 = \langle V^2(t)/2 \rangle^2$ (see Davenport and Root, 1958).

Here no distinction is made between the noise originating within the instrument and the external noise.

2.2 TOTAL POWER BEAMS

To form a total power beam voltages from all N elements, with appropriate phase distributions, are added together and passed through a square law detector.

2.2.1 Central Beam

The noise output in the central beam is given by

$$[\sum_{i=1}^{N} V_i \cos(\omega_c t+\phi_i)]^2 .$$

Here V_i and ϕ_i imply $V_i(t)$ and $\phi_i(t)$ respectively with i as a running index to represent different elements.

Noise output can be written as

$$\sum_{i=1}^{N} V_i \cos(\omega_c t+\phi_i) \cdot \sum_{j=1}^{N} V_j \cos(\omega_c t+\phi_j)$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{2} V_i V_j (\cos(2\omega_c t+\phi_i+\phi_j)+\cos(\phi_i-\phi_j)) .$$

Time average of the 1st set of terms inside the summation is zero over a cycle of $\omega_c t=2\pi$, during which time V's and ϕ's can be assumed to be unchanged. Thus the contributions from only the 2nd set of terms is relevant. Thus the noise output

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} V_i\ V_j\ \cos(\phi_i-\phi_j)$$

$$= \frac{1}{2}\sum_{i=1}^{N} V_i^{\,2} + \sum_{i=1}^{N-1}\sum_{j>i} V_i\ V_j\ \cos(\phi_i-\phi_j).$$

All V's and ϕ's are variables on time scales ~ $1/\Delta\nu$, hence the noise output will consist of independent peaks, both positive and negative about the mean, at intervals of duration ~ $1/\Delta\nu$.

2.2.2 Neighbouring Beam

Due to the introduction of instrumental phase delay, $\Delta\phi$, between adjacent elements, the noise output for the neighbouring beam can be written as

$$\left[\sum_{i=1}^{N} V_i\ \cos(\omega_c t+\phi_i+(i-1)\Delta\phi)\right]^2$$

$$= \sum_{i=1}^{N} V_i\ \cos(\omega_c t+\phi_i+(i-1)\Delta\phi)\ \cdot \sum_{j=1}^{N} V_j\ \cos(\omega_c t+\phi_j+(j-1)\Delta\phi)$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} V_iV_j\ \cos(\phi_i-\phi_j+(i-j)\Delta\phi) \quad \text{...(ignoring the high frequency terms)}$$

$$= \frac{1}{2}\sum_{i=1}^{N} V_i^{\,2} + \sum_{i=1}^{N-1}\sum_{j>i} V_iV_j\ \cos(\phi_i-\phi_j+(i-j)\Delta\phi)\quad .$$

2.2.3 Variance Of Noise In Individual Beams

Various terms within the expressions for noise output are uncorrelated or linearly independent, so total variance is the sum of variances for the individual terms.

Terms $V_iV_j\ \cos(\phi_i-\phi_j)$ have a zero mean,

$$\Rightarrow \text{ Variance of } \left(\sum_{i=1}^{N-1} \sum_{j>i} V_i V_j \cos(\phi_i - \phi_j)\right)$$

$$= \left< \sum_{i=1}^{N-1} \sum_{j>i} V_i V_j \cos(\phi_i - \phi_j) \cdot \sum_{k=1}^{N-1} \sum_{\ell>k} V_k V_\ell \cos(\phi_k - \phi_\ell) \right>$$

$$= \frac{1}{2} \sum_{i=1}^{N-1} \sum_{j>i} \sum_{k=1}^{N-1} \sum_{\ell>k} \left< V_i V_j V_k V_\ell \left(\cos(\phi_i - \phi_k - \phi_j + \phi_\ell) + \cos(\phi_i + \phi_k - \phi_j - \phi_\ell)\right) \right>.$$

The second set of terms inside the summation will average to zero, while in the first set only the terms with $\left\{ \begin{matrix} i=k \\ j=\ell \end{matrix} \right\}$ will have finite time averages.

Thus variance of $\left(\sum_{i=1}^{N-1} \sum_{j>i} V_i V_j \cos(\phi_i - \phi_j) \right)$

$$= \frac{1}{2} \sum_{i=1}^{N-1} \sum_{j>i} \langle V_i^2 V_j^2 \rangle = \frac{1}{2} \sum_{i=1}^{N-1} \sum_{j>i} \langle V_i^2 \rangle \langle V_j^2 \rangle,$$

the last step is allowed because V_i and V_j, for $i \neq j$, are statistically independent.

As mentioned in Section 2.1, variance of $(V_i^2/2) = \langle V_i^2/2 \rangle^2$.

Thus variance of noise, σ_1^2, in Central Beam

$$= \sum_{i=1}^{N} \langle V_i^2/2 \rangle^2 + 2 \sum_{i=1}^{N-1} \sum_{j>i} \langle V_i^2/2 \rangle \langle V_j^2/2 \rangle$$

$$\text{or} \quad \sigma_1^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} \langle V_i^2/2 \rangle \langle V_j^2/2 \rangle .$$

Similarly we can calculate variance $\sigma_1'^2$ for the neighbouring beam and we find

$$\sigma_1'^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} \langle V_i^2/2 \rangle \langle V_j^2/2 \rangle = \sigma_1^2 .$$

$$\therefore \quad \sigma_1 \sigma_1' = \sigma_1^2 .$$

In case all elements are identical with the same average noise power

$(V_0^2/2) = \langle V_i^2/2 \rangle$ for all i,

then $\sigma_1^2 = N^2(V_0^2/2)^2$, or $\sigma_1 = N(V_0^2/2)$,

which is N times the noise power from an individual element.

2.2.4 Covariance Of Noise In The Central And Neighbouring Beams

Representing the covariance as C_1, we note that the noise output in both the central and the neighbouring beams contain the identical terms $\frac{1}{2}\sum V_i^2$ with non-zero mean. Contributions of these terms to the covariance is $\sum_i \langle V_i^2/2 \rangle^2$. Contribution of the remaining terms to the covariance is given by

$$\left\langle \sum_{i=1}^{N-1} \sum_{j>i} V_i V_j \cos(\phi_i - \phi_j) \cdot \sum_{k=1}^{N-1} \sum_{\ell>k} V_k V_\ell \cos(\phi_k - \phi_\ell + (k-\ell)\Delta\phi) \right\rangle,$$

where the only non-zero average terms with $\left\{ \begin{smallmatrix} i=k \\ j=\ell \end{smallmatrix} \right\}$ are

$$\frac{1}{2} \sum_{i=1}^{N-1} \sum_{j>i} \langle V_i^2 V_j^2 \rangle \cos((i-j)\Delta\phi).$$

Thus

$$C_1 = \sum_{i=1}^{N} \langle V_i^2/2 \rangle^2 + 2 \sum_{i=1}^{N-1} \sum_{j>i} \langle V_i^2/2 \rangle \langle V_j^2/2 \rangle \cos((i-j)\Delta\phi)$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} \langle V_i^2/2 \rangle \langle V_j^2/2 \rangle \cos((i-j)\Delta\phi) .$$

Now Normalized Covariance or the Correlation Coefficient is defined as

$\rho_1 = C_1/(\sigma_1 \sigma'_1)$. Thus

$$\rho_1 = \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} \langle V_i^2/2 \rangle \langle V_j^2/2 \rangle \cos((i-j)\Delta\phi)}{\sum_{i=1}^{N} \sum_{j=1}^{N} \langle V_i^2/2 \rangle \langle V_j^2/2 \rangle} .$$

In case of N identical elements, we simplify it to get

$$\rho_1 = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \cos((i-j)\Delta\phi) = \left[\frac{\sin(N\Delta\phi/2)}{N \sin(\Delta\phi/2)} \right]^2$$

Which is the same as the normalized Total power beam pattern as a function of $\Delta\phi$ (Appendix A.1)

2.3 CORRELATOR BEAMS

As stated earlier, to form a correlator beam voltages from the north-arm and the south-arm are multiplied together.

2.3.1 Central Beam

We write the sum of the voltages in the north-arm as $\sum V_i \cos(\omega_c t+\phi_i)$, while to avoid any ambiguity we write the sum as $\sum V'_j \cos(\omega_c t+\phi'_j)$ for the south-arm. In whole of the Section 2.3, indices i and k will be used only for elements in the north-arm, while j and ℓ will be used for those in the south-arm. Then the noise output is given by

$$\sum_{i=1}^{n} V_i \cos(\omega_c t+\phi_i) \cdot \sum_{j=1}^{m} V'_j \cos(\omega_c t+\phi'_j)$$

$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} V_i V'_j \cos(\phi_i-\phi'_j) \quad \text{... (ignoring the high frequency terms) .}$$

This noise output, which has a zero mean, will consist of independent peaks (both positive and negative) at intervals of duration $\sim 1/\Delta\nu$.

2.3.2 Neighbouring Beam

We write the noise output as

$$\sum_{i=1}^{n} V_i \cos(\omega_c t+\phi_i + \frac{2i-1}{2}\Delta\phi) \cdot \sum_{j=1}^{m} V'_j \cos(\omega_c t+\phi'_j - \frac{2j-1}{2}\Delta\phi) .$$

Here we have chosen the mid-point in between the north and the south arms as the phase-centre for computational convenience.

Again ignoring the high frequency terms,

$$\text{noise output} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} V_i V_j' \cos((\phi_i-\phi_j')+(i+j-1)\Delta\phi).$$

2.3.3 Variance Of Noise In Individual Beams

Variance of noise in the central beam can be written as

$$\sigma_2{}^2 = \frac{1}{4} < \sum_{i=1}^{n}\sum_{j=1}^{m} V_i V_j' \cos(\phi_i-\phi_j') . \sum_{k=1}^{n}\sum_{\ell=1}^{m} V_k V_\ell' \cos(\phi_k-\phi_\ell') >$$

$$= \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{\ell=1}^{m} < V_i V_j' V_k V_\ell' \cos(\phi_i-\phi_j') \cos(\phi_k-\phi_\ell') >.$$

Time average of only the terms with $\{ \begin{matrix} i=k \\ j=\ell \end{matrix} \}$ will be finite, all other terms will average to zero.

$$\therefore \quad \sigma_2{}^2 = \frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{1}{2} < V_i{}^2 V_j'^2 > = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} <V_i{}^2/2> <V_j'^2/2> .$$

Similarly we can calculate variance $\sigma_2'^2$ for the neighbouring beam and we get

$$\sigma_2'^2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} <V_i{}^2/2> <V_j'^2/2> = \sigma_2{}^2 .$$

$$\therefore \quad \sigma_2 \, \sigma_2' = \sigma_2{}^2 .$$

In case all n elements in the north-arm are identical with the same noise power $V_0{}^2/2 = <V_i{}^2/2>$ for all i, and in the same way all m south elements have the same noise power $V_0'^2/2 = <V_j'^2/2>$ for all j, we can write

$$\sigma_2'^2 = \sigma_2{}^2 = \frac{1}{2}\, nm\, (V_0{}^2/2)\ (V_0'^2/2) .$$

2.3.4 Covariance Of Noise In The Central And Neighbouring Beams

We represent the covariance by C_2, then

$$C_2 = \frac{1}{4} < \sum_{i=1}^{n} \sum_{j=1}^{m} V_i V'_j \cos(\phi_i - \phi'_j) \cdot \sum_{k=1}^{n} \sum_{\ell=1}^{m} V_k V'_\ell \cos(\phi_k - \phi'_\ell + (k+\ell-1)\Delta\phi) >.$$

Here again only the terms with $\{ \begin{matrix} i=k \\ j=\ell \end{matrix} \}$ will have finite contribution.

$$\therefore C_2 = \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{1}{2} <V_i{}^2 V'_j{}^2> \cos((i+j-1)\Delta\phi)$$

$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} <V_i{}^2/2> <V'_j{}^2/2> \cos((i+j-1)\Delta\phi) \ .$$

The Normalized Covariance or the Correlation Coefficient, $\rho_2 = C_2/(\sigma_2 \sigma'_2)$, is given by

$$\rho_2 = \frac{\sum_{i=1}^{n} \sum_{j=1}^{m} <V_i{}^2/2> <V'_j{}^2/2> \cos((i+j-1)\Delta\phi)}{\sum_{i=1}^{n} \sum_{j=1}^{m} <V_i{}^2/2> <V'_j{}^2/2>} \ .$$

Using $V_0{}^2/2$ and $V'_0{}^2/2$ for the north and south elements respectively, we can simplify the expression for ρ_2 as

$$\rho_2 = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \cos((i+j-1)\Delta\phi)$$

$$= \frac{\sin(n\Delta\phi/2)}{n \sin(\Delta\phi/2)} \ \frac{\sin(m\Delta\phi/2)}{m \sin(\Delta\phi/2)} \ \cos\left(\frac{n+m}{2} \Delta\phi\right) ,$$

which is the same as the normalized correlator beam pattern as a function of $\Delta\phi$ (Appendix A.2).

2.4 EFFECT OF POST-DETECTION TIME CONSTANT

In all our discussions, we have ignored the effect of any post detection time constant, and calculated variances and covariances as if the output were unsmoothed. If the outputs were smoothed with a time constant

τ before the variances and covariances are calculated, we would expect these to be uniformly less by a factor $\Delta\nu\tau$. But the normalized covariances, i.e. correlation coefficients, will not be affected by the smoothening. We can see it in another way.

Consider a large number N of independent samples of a pair of random variables X and Y each, then correlation coefficient, ρ_1, is defined as

$$\rho_1 = \frac{\sum_{i=1}^{N} X_i Y_i}{\left[\sum_{i=1}^{N} X_i^2 \sum_{j=1}^{N} Y_j^2\right]^{1/2}}$$

We assume that both X and Y each have a zero mean.

Now we form another pair of variables x and y, obtained from X and Y, by taking a running mean (or even block mean) of m points each, so that

$$x_i = \frac{1}{m}\sum_{j=1}^{m} X_{i+j-1} \quad \text{and } y_i = \frac{1}{m}\sum_{j=1}^{m} Y_{i+j-1},$$

both x and y each will have a zero mean.

Now correlation co-efficient of x and y can be written as

$$\rho_2 = \frac{\sum_{i=1}^{N-m+1} x_i y_i}{\left[\sum_{i=1}^{N-m+1} x_i^2 \sum_{j=1}^{N-m+1} y_j^2\right]^{1/2}}, \quad \text{we assume } N/m \gg 1.$$

Now substituting x_i and y_i in terms of X_i and Y_i in ρ_2, it is easily verified that $\rho_2 = \rho_1$. Thus a constant linear correlation between the sets of values of two variables cannot be destroyed by a mere smoothening. So a finite time constant will have no effect upon our final results.

Fig. 2.2 shows correlation coefficient of the actually observed noise as a function of beam separation for both the total power beams with N=22,

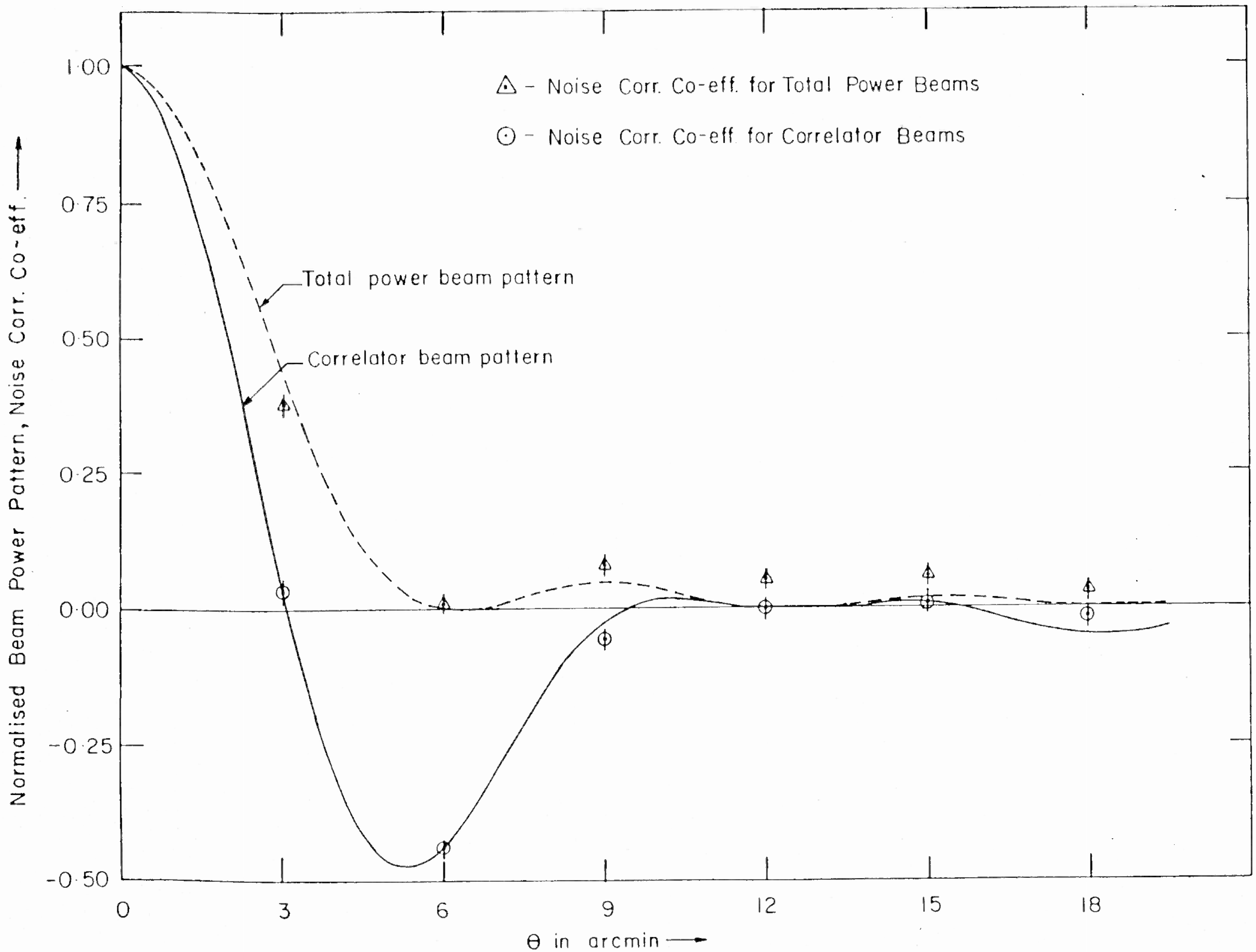

Fig 2.2 Noise correlation coefficient as a function of the angle of the beam separation and the angular power pattern for the beam

and the correlator beams with n=m=11, of the Ooty Radio Telescope. Here RC time constants of 3 msec and 500 msec were used for the total power and correlator beams respectively. The match with the beam pattern is very good in both cases. Errors in the individual correlation coefficient points are due to the finite number of sample points, a total of 1000 sample points for either beam were used in all cases.

2.5 SOME GENERALIZATIONS AND CONCLUSIONS

In our discussions above we calculated the correlation coefficient between the 'Central beam' and a 'neighbouring beam'. Actually in our calculations we can replace the central beam by a beam loooking towards an arbitrary direction, and our results still remain the same, but $\Delta\phi$ in the expressions for correlation coefficient will then represent the differential phase gradient corresponding to the angle of separation between this beam and the neighbouring beam. We assume throughout that our beams are narrow and that accordingly the angle of separation, of our interest, between two beams is also small.

Now suppose the separation between successive elements varies along the array. Then the arguments of $\cos((i-j)\Delta\phi)$ and $\cos((i+j-1)\Delta\phi)$ in the expressions for correlation coefficient for both the total power beams and the correlator beams will get replaced by the appropriate values, in general $\Delta\phi$ will be a function of both i and j. But the expressions for beam patterns also will get modified in exactly the same manner as the expressions for the correlation coefficient. We could also change the combination of elements forming the beams, especially in the case of correlator beams. Here again expressions for the correlation coefficient and the beam pattern get altered in an identical manner.

Let us consider the case when all antenna elements are not of an identical nature. We have already seen that in the case of correlator beams, if the individual elements in the north-arm are not similar to those in the south-arm, our results still hold true. As a general rule we can state that our results hold true for the following case: If there are two sets of elements and the combined voltage outputs of one set of elements are being multiplied by the combined voltage outputs of the other set of elements during beam formation, then the elements in one set need not be similar to those in the other set. But the elements within each set whose voltage outputs are being added should not all be of arbitrary nature, the ratio between the output noise power and the voltage response for a unit point source should be same for all elements within the set. The last condition is not really a very restrictive one, it concludes the case of identical elements within the set.

Thus we see that under very general configurations of beam formation systems, the correlation co-efficient of noise in two simultaneous beams, formed from a common set of antenna elements, as a function of the angle of separation of the two beams is identical to the individual beam pattern. This result can be used to estimate the expected improvement in signal to noise ratio whenever signal from a common source appears in two or more beams simultaneously. Such a premonition is useful as well in cases where no improvement is expected, as a lot of wasteful effort can be avoided.

Fig. 2.1 shows the actual occultation (Immersion) observations of a radio source, whose response appeared simultaneously in more than one beam of ORT. Only six of the total 12 beams, i.e. beam 7 to 12, are shown here; the individual beams marked by 7, 8, etc. The radio source seems to lie almost half-way between 9th and 10th beams, as the response is almost the same in these two beams. As the noise correlation between two neighbouring ORT correlator beams, separated by 3 arcmin, is almost zero (see fig.2.2), the signal to noise ratio would improve by a factor of √2 if beams 9 and 10 are combined here. But no such further improvement in signal to noise ratio will result if the negative signal present in beam 8 or beam 11 (which is about 50% of that in beam 9 or beam 10) is also sought to be combined. The reason being that there is an anticorrelation of noise at about 50% level (see fig. 2.2) between beams 8 and 10 (as well as between beams 9 and 11), which are at a separation of 6 arcmin. Such an information is very useful in utilizing the best possible sensitivity of a multiple-beam phased array system.