CHAPTER 2

THEORY OF INTERPLANETARY SCINTILLATIONS

2.1 Introduction

In this chapter we will derive various theoretical relationships which will enable us to estimate, from the observed intensity fluctuations, the properties of the IP medium and the structure of the scintillating source. The observed intensity fluctuations arise because of scattering of radio waves in the IP medium. The IP medium, which is in a plasma state, has a refractive index n given by

$$n^2 = 1 - \nu_0^2/\nu^2$$

where ν_0 is the plasma frequency, given by

$$\nu_0 = \sqrt{Ne^2/\pi m} \approx 9000\sqrt{N} \quad \mathrm{Hz}$$

where N in the number of electrons per cubic centimetre and e and m are the charge and mass of the electron. The density of the IP plasma is not uniform but varies irregularly with position and thus the refractive index of the medium also changes irregularly with position. It is these fluctuations in the refractive index of the medium that scatter radio waves and produce intensity fluctuations. To derive an expression relating the observed intensity pattern on the ground to the refractive index fluctuations in the medium we have to solve the wave equation in the IP medium. This presents considerable problems, since for arbitrary functional forms for the refractive index variations the wave equation can be solved only under very restricted conditions. Even if the wave equation could be solved, the solution would not be of much use since the variation in the refractive index are so complicated that an exact description of it is of little value.

What is more useful is a statistical description of the fluctuations in the medium, in terms of their correlation functions and probability distributions. Thus what we require from the theory are relationships between the correlation functions of the intensity fluctuations on the ground and the correlation functions of the refractive index fluctuations in the medium. To estimate the auto correlation function (acf) of the intensity fluctuations on the ground we have to estimate the intensity at two different points on the ground, take their product and find the average value of this product.

The problem is of considerable complexity and two approaches have been developed based largely on the actual physical situation encountered. In optical scintillations where the observer is located almost in the random medium, the direct solution of the wave equation has been attempted by making various approximations. However in interplanetary and ionospheric scintillation where the random medium is fairly localised and the observer is located at a large distance from this region, the phase screen model has been developed, in which the random medium is replaced by a thin screen which impresses on the incident wavefront the same phase fluctuations that the medium would have produced. Once the properties of the phase screen are specified, the problem reduces to one of diffraction in free space, which, as will be shown, is fairly tractable. The only problem in this approach is to relate the properties of the screen to the actual properties of the medium. For this one has to go back and try to solve the wave equation in the random medium. Inspite of this, the thin screen model is able to provide results under a number of conditions and we will be concerned with this model in most of our discussions. However before going into the thin screen model, we will, in the next section, give a brief discussion of the different approximations that have been used for studying the propagation of waves in a randomly inhomogeneous medium.

In section 2.3 we will discuss the use and validity of the thin screen approximation in the study of IPS. Even with the thin screen approximation it is not always possible to derive simple mathematical relationships between the properties on the ground and those on the screen. While numerical calculations are possible, and have been made, analytical expressions can be derived only if we make suitable approximations. Three approximations that have been made are the Weak scattering, Geometrical optics and the Far field approximations. The derivation of these approximate results have been made by different workers who have used different methods to arrive at the final expressions. In section 2.4 we study the thin screen model in detail and derive an expression for the spatial power spectrum of the intensity fluctuations which is in the form of an integral of a function which contains the correlation functions of the phase fluctuations on the screen. The integration cannot, in general, be performed, but, as we will show in the subsequent sections, the integral can be used to derive in a simple and straight forward fashion the three approximations mentioned earlier and the formulae used for numerical computations. In section 2.9 and 2.10 we examine how the observed intensity fluctuations are affected by the bandwidth of the observation and the finite size of the scintillating source. In section 2.11 we see how the temporal intensity fluctuations are related to the spatial fluctuations when the screen is moving with respect to the observer. The cases of a frozen in pattern on the screen and of a pattern that evolves as it drifts, are considered and it is shown that in the latter case the lifetime of the intensity pattern on the ground is, in the strong scattering regime, much smaller than the life-time of the phase pattern on the screen. The consequences of this on various observations are discussed. In the last section we summarise the various theoretical results.

In this chapter we will not discuss the angular broadening of radio sources due to scattering in an inhomogeneous medium. A good discussion of the problem can be found in Ratcliffe (1956). A result we will frequently use is that for small angles, the angular spectrum that is produced when a plane wave passes through a thin screen which imposes random phase fluctuations, is given by the Fourier transform of the autocorrelation function of the electric field on the screen, with distance measured in units of λ , the wavelength. In section 2.4 we will derive an expression for the acf and show that the correlation length is equal to a, the scale size of the phase fluctuations, if the r.m.s. phase fluctuation, $\phi_0 \ll 1$ and equal to a/ϕ_0, if $\phi_0 \gg 1$. Thus the width of the angular spectrum, which is the typical scattering angle, θ_{scat}, is given by

$$\theta_{scat} = \lambda/2\pi a \qquad \text{if } \phi_0 \ll 1$$

$$\theta_{scat} = \lambda\phi_0/2\pi a \qquad \text{if } \phi_0 \gg 1$$

It can be shown (Ratcliffe (1956)) that the autocorrelation function of the electric field across the wavefront does not change as the wavefront propagates, and so the scale size of the electric field at the plane of the observer is given by

$$\ell_E \simeq a \qquad \phi_o \ll 1$$

$$\ell_E \simeq a/\phi_o \qquad \phi_o \gg 1$$

We will frequently use these results in the discussions that follow.

2.2 Wave Propagation in a Random Medium

Wave propagation in a medium whose refractive index varies irregularly with position has been studied in connection with a number of different problems like atmospheric scintillation at optical frequencies, and ionospheric and interplanetary scintillations at radio frequencies. A considerable body of literature exists on the subject and good discussions can be found in Chernov (1960), Tatarski (1961), Frisch (1968), Strohbehn (1968) and Barabanenkov et al. (1971). In this chapter we will briefly touch upon various aspects of the problem and derive some results that are of use in interpreting the observations.

We consider plane monochromatic electromagnetic waves travelling in the positive z direction and incident on a region where the refractive index n varies irregularly with position. The irregular medium extends from $\pm L/2$ in the z direction and $\pm\infty$ in the x and y directions. The irregular variation of the refractive index is such a complicated function of position that the situation can be described only satistically. We will separate the mean and fluctuating parts of the refractive index as

$$n^2(\underline{r}) = n_0^2 + \mu(\underline{r})$$

and regard $\mu(\underline{r})$ to be a homogeneous isotropic random variable whose acf is defined as

$$\mu_0^2 \rho_r(\underline{r}) = \langle \mu(\underline{r}_1) \mu(\underline{r}_1 + \underline{r}) \rangle$$

where the angular bracket stands for averaging over space, which, with the assumption of ergodicity, is equivalent to ensemble averaging. The problem is to express the electric field, intensity and their correlation functions at any plane z=Z, which can be either inside or outside the medium, in terms of $\mu(\underline{r})$ and $\rho_r(\underline{r})$.

To estimate the electric field at any point in the medium we have to solve the wave equation in the medium. In all our discussions we will assume that the variation of the refractive index about the mean value is small i.e. $\mu_0 \ll 1$ and that the scale size of the irregularities is much larger than the wavelength of the observations. Both these assumptions are quite justified for the IP medium. We will also assume that the mean electron density is small, so that $n_0 \simeq 1$. Since the scale size $a \gg \lambda$ we can describe the electric field of the waves in the random medium by the scalar wave equation which is

$$\nabla^2 E(\underline{r}) + k^2 n^2(\underline{r}) E(\underline{r}) = 0 \tag{2.1}$$

where $k = 2\pi/\lambda$

Exact solution of partial differential equations with random coefficients is not possible and so one is forced to make approximations. Three methods commonly used are the

a) Geometrical optics approximation

b) Born approximation

c) Method of smooth perturbation.

2.2a Geometrical Optics Approximation

This is the short wave length approximation and is useful when the wavelength is much shorter than the scale size of the irregularities, as is the case in the IP medium, so that wave effects like interference and diffraction can be neglected. This approximation has been extensively used in ionospheric and interplanetary scintillation and is used in most of the thin phase screen models to relate the properties of the screen to those of the medium.

Using the identity

$$\nabla^2 \log u \equiv \frac{\nabla^2 u}{u} - (\nabla \log u)^2$$

the wave equation (2.1) can be written as

$$\nabla^2 \log E(\underline{r}) + (\nabla \log E(\underline{r}))^2 + k^2 n^2(\underline{r}) = 0 \qquad (2.2)$$

If we express E in the form

$$E(\underline{r}) = A(\underline{r})\, e^{i\phi(\underline{r})}$$

and substitute this in equation (2.2), we get, by equating the real and imaginary parts,

$$\nabla^2 \log A + (\nabla \log A)^2 - (\nabla \phi)^2 + k^2 n^2 = 0 \qquad (2.3)$$

$$\nabla^2 \phi - 2 \nabla \log A \cdot \nabla \phi = 0 \qquad (2.4)$$

For plane waves the electric field varies roughly as $e^{i\underline{k}\cdot\underline{r}}$. So $|\nabla\phi| \approx \nabla \underline{k}\cdot\underline{r}$ is of the order of k. However the amplitude A, which varies because of the perturbations in the medium, cannot change appreciably in distances smaller than the scale size of the irregularities. Therefore

$$\nabla^2 \log A + (\nabla \log A)^2 = \frac{\nabla^2 A}{A} \tag{2.5}$$

is of the order of $1/a^2$. So if the condition $\lambda \ll a$ is satisfied, we can neglect the first two terms in equation (2.3), which reduces to

$$(\nabla \phi)^2 = k^2 n^2 \tag{2.6}$$

Equation (2.4) and (2.6) form the basic equations of Geometrical optics. Equation (2.6) can be solved by integrating the phase along the trajectory of the ray $\underline{\gamma}(\sigma)$ giving

$$\phi(\underline{\gamma}(s)) = \frac{2\pi}{\lambda} \int_0^s n(\underline{\gamma}(\sigma))\, d\sigma$$

where s denotes the arc length along the ray and the ray trajectory is given by (see Frisch (1968))

$$\frac{d}{ds}\left(n(\underline{\gamma}(s)) \frac{d}{ds}(\underline{\gamma}(s))\right) = \nabla\left(n(\underline{\gamma}(s))\right) \tag{2.7}$$

While equations (2.6) and (2.7) can be solved for non random media by using numerical techniques, these equations can seldom be used for random media because of their complexity. What is usually done is to solve equations (2.4) and (2.6) by perturbation techniques.

We expand A and ϕ as a series in ϵ where ϵ is a parameter indicating the smallness of the perturbation. We set

$$\Phi = \Phi_0 + \sum_{i=1}^{\infty} \epsilon^i \Phi_i$$

$$\log A = \log A_0 + \sum_{i=1}^{\infty} \epsilon^i A_i$$

$$n(\underline{r}) = n_0^2 + \epsilon \mu(\underline{r}) = 1 + \epsilon \mu(\underline{r})$$

Substituting these in equation (2.6) and collecting terms of equal powers of ϵ we get

$$(\nabla \phi_0)^2 = k^2$$

$$2 \nabla \phi_\mu \cdot \nabla \phi_1 = k^2 \mu(\underline{r}) \tag{2.8}$$

$$2 \nabla \phi_u \cdot \nabla \phi_2 = -(\nabla \phi_1)^2 \tag{2.9}$$

Since we are considering plane waves propagating in the z direction the unperturbed phase

$$\phi_u = k z$$

so that equation (2.8) becomes

$$2 k \frac{\partial \phi_1}{\partial z} = k^2 \mu(\underline{r})$$

giving

$$\phi_1(x, y, L/2) = k/2 \int_{-L/2}^{L/2} \mu(x, y, z)\, dz \tag{2.10}$$

This is the expression generally used to relate the phase fluctuations in the thin screen to the density fluctuations in the medium. We have an extra 2 in the denominator which has come because we have taken $\mu(\underline{r})$ to be the fluctuating part of n^2 and not of n.

For most applications of the thin screen model it is enough if we know the two dimensional acf of the phase fluctuations across the x-y plane. This is given by

$$\phi_0^2 \rho(u, v) = \langle \phi_1(x, y, z = L/2)\, \phi_1(x+u, y+v, z = L/2) \rangle$$

$$= \frac{k^2}{4} \iint_{-L/2}^{L/2} \langle \mu(x, y, z')\, \mu(x+u, y+v, z'') \rangle\, dz'\, dz''$$

$$= \frac{k^2 \mu_0^2}{4} \iint_{-L/2}^{L/2} \rho_n(u, v, z' - z'') \, dz' dz''$$

where we have used the definition of the acf of the refractive index variations. Changing the variables of integration to $\zeta = z' - z''$ and $\zeta' = z''$, the expression becomes

$$\phi_0^2 \rho(u, v) = \frac{k^2 \mu_0^2}{4} \int_{-L/2}^{+L/2} d\zeta' \int_{-L/2 - \zeta'}^{L/2 - \zeta'} d\zeta \, \rho_n(u, v, \zeta)$$

Since $\rho_n(u, v, \zeta)$ goes to zero when ζ is much larger than the correlation distance a, we can, if $L \gg a$, replace the limits of the ζ integral by $\pm \infty$. The fractional error we make is of the order of a/L which is negligible when $L \gg a$. With this, the phase acf becomes

$$\phi_0^2 \rho(u, v) = \frac{k^2 \mu_0^2 L}{4} \int_{-\infty}^{+\infty} \rho_n(u, v, \zeta) \, d\zeta \qquad (2.11)$$

The value of the integral is roughly equal to the scale size a, but its exact value depends on the actual shape of the acf. If we assume that the acf is a Gaussian of the form $e^{-r^2/2a^2}$, we can readily perform the integration over ζ and we get

$$\phi_0^2 \rho_n(u, v) = \frac{k^2 \mu_0^2 L a}{4} \sqrt{2\pi} \, \exp - \{(u^2 + v^2)/2a^2\} \qquad (2.12)$$

The parameter ϕ_0 which is the r.m.s. phase fluctuation produced by the medium, plays an important role in all the approximate solutions of the wave equation. For the Gaussian acf ϕ_0 is given by

$$\phi_0^2 = \sqrt{2\pi}\ \frac{k^2 \mu_0^2 L a}{4} = \sqrt{2\pi}\left(\frac{e^2}{mc^2}\right)^2 \Delta N^2 \lambda^2 L a \qquad (2.13)$$

where ΔN is the r.m.s. electron density fluctuation.

ϕ_1 is just the first term of the perturbation expansion of ϕ and it will be a good approximation for the phase fluctuations only if the higher order terms are negligible compared to ϕ_1. To examine the condition under which this is true, we have to solve equation (2.9) for ϕ_2. With $\phi_u = kz$ equation (2.9) reduces to

$$2k\ \frac{\partial \phi_2}{\partial z} = -(\nabla \phi_1)^2$$

giving

$$\phi_2 = -\frac{1}{2k}\int_{-L/2}^{L/2} (\nabla \phi_1)^2\, dz$$

An order of magnitude estimate of ϕ_2 can be got by putting $\nabla \phi_1 \approx \phi_0/a$ so that

$$|\phi_2| \approx L\phi_0^2/2ka^2$$

or $$|\phi_2/\phi_1| \approx L\phi_0/2ka^2$$

For the perturbation expansion with only ϕ_u and ϕ_1 to be a good approximation, we must have

$$|\phi_2/\phi_1| \approx L\phi_0/2ka^2 \approx (8\pi)^{1/4}\ \frac{\mu_0}{4}\left(\frac{L}{a}\right)^{3/2} \ll 1 \qquad (2.14)$$

If the thickness of the medium is so large that the inequality (2.14) is violated, we have to retain more terms in the perturbation series.

Similarly, one can write down the perturbation expansion for the amplitude which gives

$$\nabla^2 \phi_u + 2 \nabla \log A_u \cdot \nabla \phi_u = 0 \qquad (2.15)$$

$$\nabla^2 \phi_1 + 2 \left(\nabla \log A_u \cdot \nabla \phi_1 + \nabla A_1 \cdot \nabla \phi_u \right) = 0 \qquad (2.16)$$

With $\phi_u = kz$ equation (2.15) becomes

$$2k \frac{\partial}{\partial z} \log A_u = 0$$

which shows that the amplitude of the unperturbed radiation is independent of z, which is as it should be. Equation (2.16) then reduces to

$$\frac{\partial A_1}{\partial z} = -\frac{1}{2k} \nabla^2 \phi_1$$

giving

$$A_1(x, y, L/2) = -\frac{1}{2k} \int_{-L/2}^{L/2} \nabla^2 \phi_1 \, dz \qquad (2.17)$$

Using ϕ_1 as given in equation (2.10) we can, in principle, solve this equation to get A_1. Our main interest in Geometrical optics is to relate the properties of the medium with the properties of the thin screen and as such our interest is not to solve equation (2.17) for A_1, but to know under what conditions the amplitude fluctuations are negligible and the phase screen approximation is adequate. A_1 is the fluctuation of the logarithm of the amplitude and A_1 and ϕ_1 can be regarded as the real and imaginary part of a complex phase. For amplitude fluctuations to be negligible we must have $|A_1/\phi_1| \ll 1$. From equation (2.17) we can get an order of magnitude estimate of A_1 by setting $\nabla^2 \phi_1 \approx \phi_0/a^2$ giving

$$|A_1/\phi_1| \approx L/2ka^2$$

Thus only when $L/2ka^2 \ll 1$ can we replace the extended medium by a thin phase screen. When $L/2ka^2 \gtrsim 1$ amplitude fluctuations develop and if we want to replace the medium by a thin screen we would require both the amplitude and the phase to vary on the screen.

In the situation where we have a slab of irregular medium of thickness L and we are interested in the field at a large distance $Z (\gg L)$ from the slab, the first order perturbation solution of the Geometrical optics equations is of limited use because it neglects diffraction and interference effects which are now important. $\mu(x,y,z)$ vanishes outside the slab and so ϕ_1 (equation (2.10)) is independent of z outside the slab. But $\nabla \phi_1$ and $\nabla^2 \phi_1$ do not vanish outside the slab and so ϕ_2 and A_1 continuously increase with z. Thus for sufficiently large z the first order perturbation theory breaks down. Solving for higher orders terms is messy and has seldom been attempted. However, if the thickness of the slab is sufficiently small so that the first order Geometrical optics solution is valid for the exit plane of the slab, then, since the further propagation is in free space, the field at any distance can be got by studying the diffraction of this perturbed wave front. Such an approach is not limited to small distance from the slab, as is the first order perturbation solution of Geometrical optics, and this is the motivation behind the thin phase screen model which will be discussed later.

2.2b Born Approximation

We can rewrite equation (2.1) as an inhomogeneous equation of the form

$$\nabla^2 E(\underline{r}) + k^2 E(\underline{r}) = - k^2 \mu(\underline{r}) E(\underline{r})$$

whose formal solution can be written as an integral equation of the form (Morse and Feshbach (1953))

$$E(\underline{r}) = E_0(\underline{r}) + \frac{k^2}{4\pi} \int \frac{e^{ik|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} \mu(\underline{r}') E(\underline{r}') d\underline{r}' \quad (2.18)$$

where $E_0(\underline{r})$ is a solution of the homogenous Helmholtz equation

$$\nabla^2 E_0(\underline{r}) + k^2 E_0(\underline{r}) = 0$$

E_0 is the field in the absence of the perturbing medium and for plane waves propagating in the z direction

$$E_0(\underline{r}) = e^{ikz}$$

Writing the differential equation in the form of the integral equation (2.18) does not solve anything since $E(\underline{r})$, which is the unknown, appears on the right hand side also. The Born approximation consists of replacing $E(\underline{r})$ inside the integral by $E_0(\underline{r})$. With this, the integral can be evaluated and $E(\underline{r})$ determined. The condition for the validity of the approximation is that $E(\underline{r})$ should not differ much from $E_0(\underline{r})$ i.e.

$$|E(\underline{r}) - E_0(\underline{r})| \ll E_0(\underline{r}) \quad (2.19)$$

If we write

$$E(\underline{r}) = E_0(\underline{r})\, e^{\{i\phi(\underline{r}) - \chi(\underline{r})\}}$$

where ϕ and χ are the phase and amplitude change produced by the medium, the condition (2.19) becomes

$$\left| e^{\{ i\phi(\underline{r}) + \chi(\underline{r}) \}} - 1 \right| \ll 1$$

or

$$\left| i\phi(\underline{r}) + \chi(\underline{r}) \right| \ll 1$$

Since the amplitude fluctuations are generally smaller than the phase fluctuation, the condition for the validity of the Born approximation reduces to $\phi_0 \ll 1$, where ϕ_0 is the r.m.s. phase fluctuation produced by the medium. Thus, so long as we are in the weak scattering regime i.e. $\phi_0 \ll 1$ we can use the Born approximation and derive results which have no restrictions on the thickness of the medium and which take into account wave effects like interference and diffraction.

2.2c The Method of Smooth Perturbation

This method has been described in considerable detail by Tatarskii (1961). We will here follow Strohbehn (1968) and give a brief description of the method. If we substitute $E(\underline{r}) = e^{\psi(\underline{r})}$ in the wave equation (2.1) we get

$$\nabla^2 \psi(\underline{r}) + \nabla\psi(\underline{r}) \cdot \nabla\psi(\underline{r}) + k^2 n^2(\underline{r}) = 0 \qquad (2.20)$$

If $\psi = \chi + iS$, χ is the logarithm of the amplitude and S is the phase. If we make a perturbation expansion of ψ and substitute

$$n^2(\underline{r}) = 1 + \epsilon\mu(\underline{r})$$

and

$$\psi(\underline{r}) = \psi_0(\underline{r}) + \epsilon\psi_1(\underline{r}) + \epsilon^2\psi_2(\underline{r}) + \cdots$$

into equation (2.20), we get, by equating equal powers of ϵ

$$\nabla^2 \psi_0 + \nabla \psi_0 \cdot \nabla \psi_0 + k^2 = 0$$

$$\nabla^2 \psi_1 + 2(\nabla \psi_0 \cdot \nabla \psi_1) = -k^2 \mu(\underline{r})$$

$$\nabla^2 \psi_2 + 2(\nabla \psi_0 \cdot \nabla \psi_2) = -\nabla \psi_1 \cdot \nabla \psi_1$$

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$$\nabla^2 \psi_n + 2(\nabla \psi_0 \cdot \nabla \psi_n) = -\sum_{j=1}^{n-1} \nabla \psi_j \cdot \nabla \psi_{n-j}$$

The first equation corresponds to the free space solution and though non linear, the equation can be readily solved. The remaining equations are of the form

$$\nabla^2 \psi_n(\underline{r}) + 2(\nabla \psi_0(\underline{r}) \cdot \nabla \psi_n(\underline{r})) = -f(\underline{r}) \qquad (2.21)$$

where $f(\underline{r})$ is known if one has solved the lower order equations. This equation can be solved by using Green's function if we make the substitution

$$\psi_n(\underline{r}) = e^{-\psi_0(\underline{r})} u(\underline{r})$$

which converts equation (2.21) to

$$\nabla^2 u(\underline{r}) + k^2 u(\underline{r}) = -e^{\psi_0(\underline{r})} f(\underline{r})$$

which has the solution

$$u(\underline{r}) = \int G(\underline{r}, \underline{r}') e^{\psi_0(\underline{r}')} f(\underline{r}') d\underline{r}'$$

where $G(\underline{r}, \underline{r}') = \dfrac{e^{ik|\underline{r}'-\underline{r}|}}{|\underline{r}'-\underline{r}|}$

Since we are considering plane waves travelling along the z direction $\psi_0(\underline{r}) = ikz$ and we get the first order perturbation term as

$$\psi_1(\underline{r}) = \frac{k}{4\pi} \int \frac{e^{ik|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} e^{-ik(z-z')} \mu(\underline{r}') d\underline{r}' \qquad (2.22)$$

In the first order approximation the electric field is

$$E(\underline{r}) = e^{ikz + \psi_1(\underline{r})} \qquad (2.23)$$

The range of ψ_1 over which this approximation is valid is a matter of some controversy. (See Barabanenkov et al. (1971)). For the validity of this perturbation expansion we must have $|\psi_2| \ll |\psi_1|$ which can be shown to be equivalent to the condition $|\psi_1| \ll 1$. This is just the condition for the validity of the Born approximation and under this condition equation (2.23) reduces to the Born approximation. However, this condition may be too rigid, and it has been shown that ψ_1 is a good approximation over the larger range

$$\langle \chi_1^2 \rangle \ll 1 \qquad \text{and} \qquad \langle (S_1(\underline{r}_1) - S_1(\underline{r}_2))^2 \rangle \ll 1$$

Thus the method of smooth perturbation is able to handle larger phase fluctuations than the Born approximation but is not valid when large amplitude fluctuations develop. The method in superior to Geometrical optics in that it takes into account wave effects and thus gives a better description of the amplitude. The phase, however, is not so sensitively dependent on wave effects and there is not much difference between the estimates of the phase in two methods.

2.3 The Thin Screen Model in IPS

When the thickness of the scattering medium is small, the electric field at any point inside the medium can be got using Geometrical optics. However, if one is interested in the field at a large distance from the medium the Geometrical optics approximation breaks down because wave effects become important. The Born approximation does take wave effects into account, but its range of validity is small, being restricted to $\phi_0 \ll 1$. A much better approximation, under these conditions, is the thin screen model in which one determines, using Geometrical optics, the field emerging from the irregular medium, and then uses the full wave equation to study the propagation of the perturbed wave front from the medium to the observer. Since the propagation from the medium to the observer is in free space, the solution of the wave equation, as will be shown in the next section, poses no great problem and useful results can be derived even for large phase and amplitude fluctuations at the ground.

In the thin screen model the irregular medium is replaced by a thin phase screen which imposes on the incident waves the same phase fluctuations that the extended medium would have produced. In the Geometrical optics approximation, the phase fluctuation imposed by the screen is related to the refractive index fluctuations in the medium by equation (2.10) which gives

$$\phi(x, y) = \frac{k}{2} \int_{-L/2}^{L/2} \mu(x, y, z)\, dz \tag{2.24}$$

If we assume that the acf of the refractive index fluctuations is a Gaussian, the acf of the phase fluctuations on the screen is (equation (2.12))

$$\phi_o^2 \rho(u,v) = \frac{k^2 \mu_o^2 La}{4} \sqrt{2\pi} \exp - \{(u^2+v^2)/2a^2\} \qquad (2.25)$$

The condition for the validity of the Geometrical optics expression (2.24) is (section 2.2a)

$$L/ka^2 \ll 1 \qquad (2.26\ a)$$

$$L\phi_o/ka^2 \ll 1 \quad \text{if} \quad \phi_o \gg 1 \qquad (2.26\ b)$$

For equation (2.25) to be valid we must have the additional condition

$$L \gg a \qquad (2.26\ c)$$

In this section we will examine the validity of these inequalities for the IP medium and see under what conditions the thin screen model can be used.

The geometry for the scattering in the IP medium is shown in figure 1. The angle, ϵ , between the line of sight to the source and the line of sight to the Sun is called the elongation. The perpendicular distance from the Sun to the line of sight to the source will be called p and this is related to ϵ by

$$p = \sin \epsilon \quad \text{A.U.} \qquad (2.27)$$

We will regard the equivalent thin screen to be located at the point where the line of sight to the source comes closest to the Sun. Thus, for any elongation, the distance z of the screen from the observer is given by

$$z = \cos \epsilon \quad \text{A.U.}$$

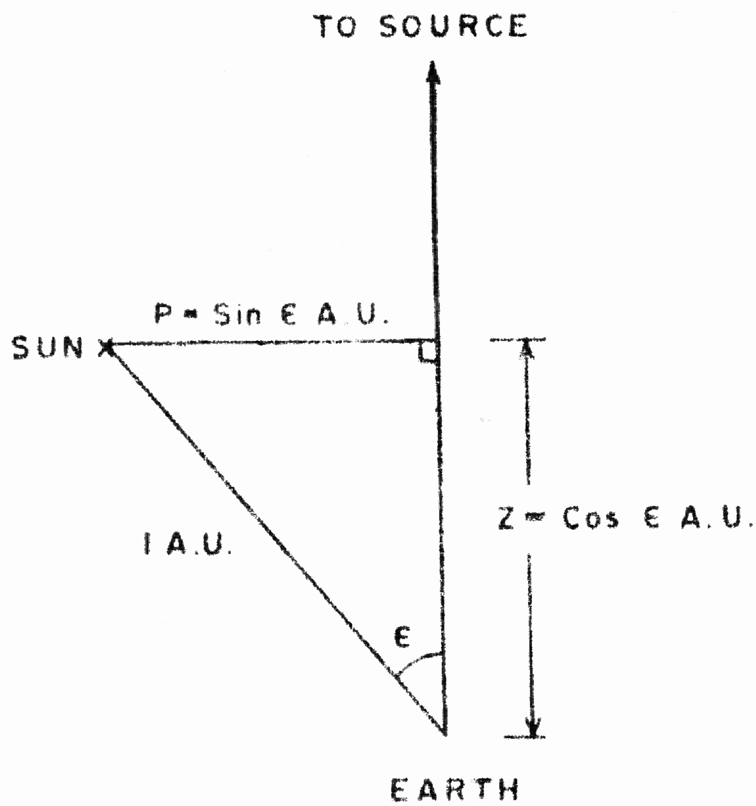

Figure 1. Geometry for scattering by the IP medium.

The IP medium is present everywhere along the line of sight to the source and the problem is to define an equivalent thickness L for the scattering region, which will enable us to check the validity of the conditions (2.26). In defining an equivalent thickness for the medium, we are facilitated by the fact that the scattering power of the IP medium decreases with distance from the Sun roughly as r^{-4} so that most of the scattering comes from those parts of the line of sight that are closest to the Sun. The thickness of the scattering region has been variously defined, but if we define it as the distance along the line of sight where the contribution to the scattering falls to half its maximum value, then, from the assumed r^{-4} dependence of the scattering power, we get the thickness as

$$L = 1.3\ p \text{ A.U.} \tag{2.28}$$

For most of the observations described here p is in the range 0.05 to 1 A.U. and since the scale size of the irregularities is in the range 10 to 100 km, the condition $L \gg a$ is always well satisfied. If we substitute for L from equation (2.28) into the condition (2.26a) and express it in practical units, the condition becomes

$$3.1 \frac{p_{A.U.}\ \lambda_{metre}}{a^2_{100\,km}} \ll 1 \tag{2.29}$$

For all the observations described here, $\lambda = 0.9$ metre and p is in the range 0.1 to 1 A.U. For $p \gtrsim 0.25$ A.U. the scale size is found to be roughly independent of elongation having a constant value of 100 km. If we substitute for a in equation (2.29) we find that the inequality is violated for $p > 0.38$ A.U. and here the thin screen model is not very good since amplitude fluctuations develop inside the medium and the first order Geometrical optics approximation for the phase does not take this into account. However, we will show that for our observations ϕ_0 goes as

$$\phi_0 = 0.04 \; p^{-1.6} \text{ radian} \tag{2.30}$$

so that for $p > 0.35$ A.U. the scattering is weak and so the Born approximation can be used to get the field at the observer. The results of the Born approximation can be got by replacing the extended medium by a number of independent thin screens at different distances from the observer and adding the fields produced by all the screens. The weakness of the scattering ensures that multiple scattering of the radiation can be neglected. Thus the results of the thin screen model are of use even for $p > 0.35$ A.U. since they offer a relatively simple way of estimating the actual field produced by the extended medium. This approach of dividing the extended into a number of screens has the advantage that intrinsic variations of the properties of the medium along the line of sight can be easily incorporated into the calculation.

Below 0.25 A.U. the scale size decreases as the line of sight approaches the Sun and in this region we will show (equation (4.32)) that

$$a = 900 \; p^{1.4} \text{ km } (p < 0.2 \text{ A.U.}) \tag{2.31}$$

Inserting equations (2.30) and (2.31) in the inequalities (2.26a,b) they become, with $\lambda = 0.9$ metre

$$0.035 \; p_{A.U.}^{-1.8} \ll 1 \tag{2.32a}$$

$$0.0014 \; p_{A.U.}^{-3.4} \ll 1 \quad \text{if} \quad \phi_0 \gg 1 \tag{2.32b}$$

Both these inequalities are violated when p is less than 0.15 A.U. and here equation (2.24) cannot be used to relate the phase fluctuation on the screen to the density fluctuations in the medium not only because amplitude fluctuations develop, but also because higher order terms in the perturbation expansion for the phase, which have been neglected in equation (2.24), now become important. If the first inequality alone were violated we could still derive useful expressions for the intensity fluctuations on the ground by using the multiple screen model described earlier, but when the second inequality is also violated this is not possible since now, the scattering is strong and multiple scattering effects have to be taken into account. Thus, though the thin screen model does give relations between the intensity fluctuations on the ground and the phase fluctuations on the screen even for $\phi_0 \gg 1$, these results cannot be used in the interpretation of the IPS observations close to the Sun at 327 MHz since for $p < 0.15$ A.U. we do not have a simple expression relating the properties of the medium to those of the screen. The evaluation of the higher order terms in the perturbation expansion for the phase and amplitude in the Geometrical optics approximation has not been attempted.

To summarise, for 327 MHz, we can, using the thin screen or multiple thin screen model, derive relations between the intensity fluctuation on the ground and the density fluctuation in the medium only for $p > 0.15$ A.U. Below 0.15 A.U. we have no adequate theory for IPS.

2.4 Mathematical Treatment of the Thin Screen Model

In this section we will develop the thin screen model in detail and derive relations between the correlation functions of the phase fluctuations on the screen and the spatial power spectra of the intensity fluctuations on the ground. We will assume that a plane monochromatic wave of unit amplitude is travelling in the positive z direction and is incident on a phase screen located at $z = 0$. The geometry is shown in figure (2). The screen extends from $\pm\infty$ in the x and y directions and imposes a phase fluctuation $\phi(x,y)$ on the incident wavefront. Since the electric field at the plane $z = 0$ is now completely specified, the field at the observers plane at a distance z, $E(\underline{\xi})$, can be written

$$E(\underline{\xi}) = \frac{k}{2\pi} \int \frac{e^{ikr}}{r} e^{i\phi(\underline{\xi}')} dS(\underline{\xi}') \tag{2.33}$$

where $\underline{\xi}$ is any point on the observer's plane, $\underline{\xi}'$ is any point on the screen, $dS(\underline{\xi}')$ is an element of area around $\underline{\xi}'$ and r is the physical distance of $dS(\underline{\xi}')$ from $\underline{\xi}$. Both $\underline{\xi}'$ and $\underline{\xi}$ are 2 dimensional vectors in the x-y plane and the distance r is given by

$$r = \sqrt{z^2 + |\underline{\xi}' - \underline{\xi}|^2}$$

Since the exact solution of equation (2.33) is not possible we will expand r as a binomial series of the form

$$r = z + \frac{|\underline{\xi}' - \underline{\xi}|^2}{2z} - \frac{1}{8}\frac{|\underline{\xi}' - \underline{\xi}|^4}{z^3} + \ldots \tag{2.34}$$

and retain only the first two terms. This approximation is valid only for $|\underline{\xi}' - \underline{\xi}|/z \ll 1$ and since, in the integral, we are going to let $\underline{\xi}'$ range from $\pm\infty$, we must examine the validity of this approximation in more detail. We will show that if the irregularities are large, so that the angle

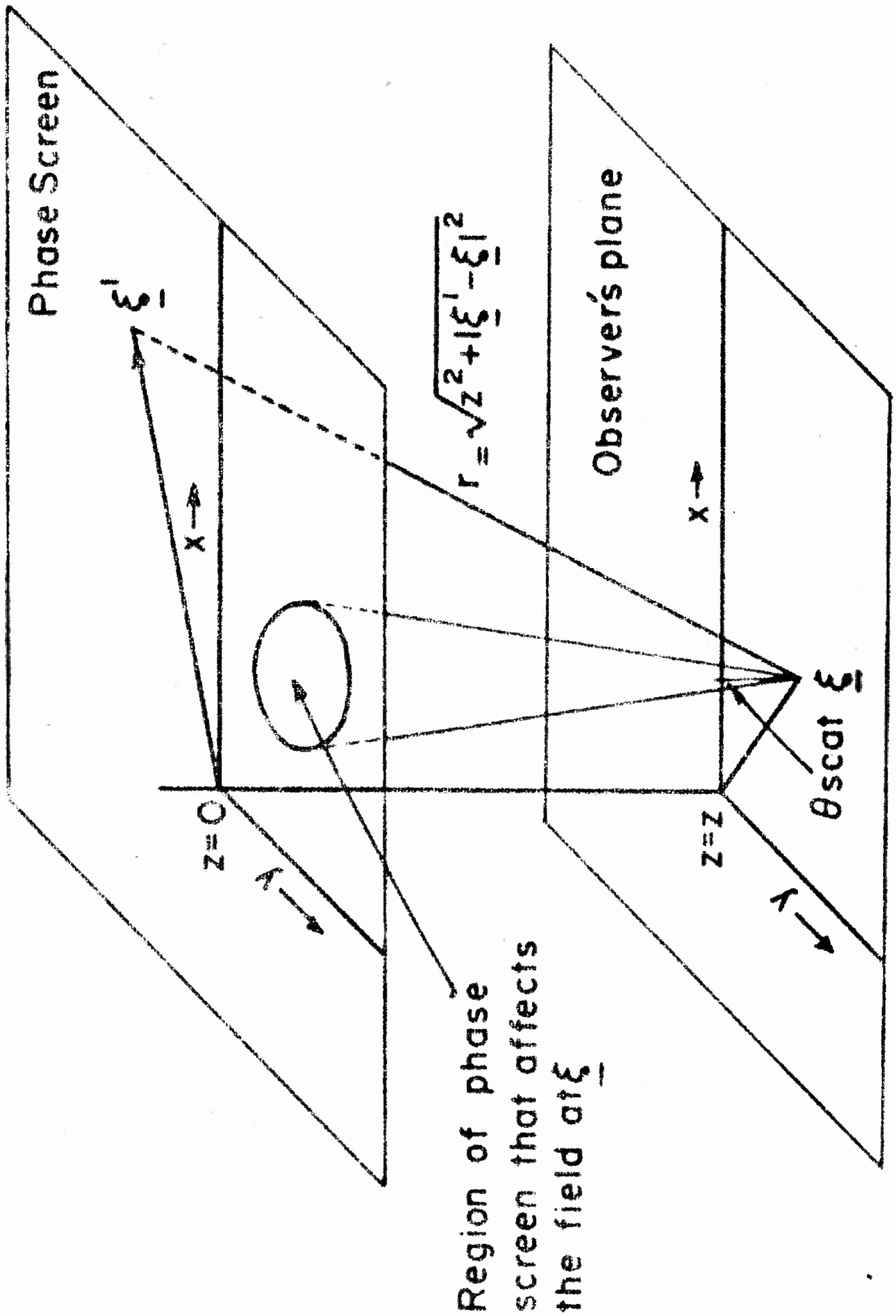

Figure 2. Geometry for the thin screen model. The field at any point ξ on the observer's plane is determined by the phase fluctuations in the circular region of the phase screen, with centre ξ and radius $Z\theta_{scat}$.

of scattering is sufficiently small, then most of the contribution to the integral in equation (2.33) comes from a small part of the screen located around $\underline{\xi}' = \underline{\xi}$ and if $\frac{|\underline{\xi}' - \underline{\xi}|}{z} \ll 1$ over this region, the approximation we are making is reasonably good.

In the ray picture the electric field at the point $\underline{\xi}$ is affected only by that portion of the screen that is within the scattering cone which is a cone with vertex at $\underline{\xi}$, axis in the z direction and semi vertical angle equal to the angle of scattering, θ_{scat}. The rays from outside this area cannot reach $\underline{\xi}$ since the scattering angle is not large enough. Since

$$\theta_{scat} = \frac{1}{ka} \qquad \phi_o \ll 1 \tag{2.35a}$$

$$= \frac{\phi_o}{ka} \qquad \phi_o \gg 1 \tag{2.35b}$$

the maximum value of $|\underline{\xi}' - \underline{\xi}|$ that is of importance

$$|\underline{\xi}' - \underline{\xi}|_{max} \simeq z\,\theta_{scat}$$

$$= z/ka \qquad \phi_o \ll 1 \tag{2.36a}$$

$$= z\phi_o/ka \qquad \phi_o \gg 1 \tag{2.36b}$$

This result can be derived directly from equation (2.33). The integrand in equation (2.33) is the product of $e^{i\phi(\underline{\xi}')}$ and e^{ikr}. If we use the binomial expansion for r

$$e^{ikr} = e^{ik\left(z + \frac{|\underline{\xi}' - \underline{\xi}|^2}{2z} + \cdots\right)}$$

$$= e^{ikz} \; e^{ik f(|\underline{\xi}' - \underline{\xi}|)\, |\underline{\xi}' - \underline{\xi}|}$$

We see that e^{ikr} behaves as a quasi sinusoidal oscillation whose frequency varies with $|\underline{\xi}' - \underline{\xi}|$. The frequency of oscillation is zero for $\underline{\xi}' = \underline{\xi}$ and it increases linearly with $|\underline{\xi}' - \underline{\xi}|$ for small values of $|\underline{\xi}' - \underline{\xi}|$. Regions of the screen at large values of $|\underline{\xi}' - \underline{\xi}|$ will not contribute appreciably to the integral because the frequency of the oscillation here will be large and there will be many oscillations before $e^{i\Phi(\underline{\xi})}$ changes appreciably which causes the integrand to average out to zero. The integral is affected by only those parts of the screen where the rate of change of the geometrical phase with respect to $\underline{\xi}'$ is small compared to the rate of change of $e^{i\Phi(\underline{\xi})}$. An estimate of the maximum value of $|\underline{\xi}' - \underline{\xi}|$ of importance can be got by setting the two rates equal which gives the equation

$$k \left. \frac{dr}{d\xi} \right|_{|\underline{\xi}' - \underline{\xi}|_{max}} \approx \frac{1}{\ell} \tag{2.37}$$

where, as a measure of the rate of variation of $e^{i\Phi(\underline{\xi})}$, we take the reciprocal of ℓ, the width of the autocorrelation function of the fluctuations of the electric field on the screen, which is

$$f_E(\underline{r}) = \left\langle \left(E(\underline{\xi}) - \bar{E}\right)\left(E(\underline{\xi} + \underline{r}) - \bar{E}\right)^* \right\rangle$$

$$= \left\langle E(\underline{\xi})\, E(\underline{\xi} + \underline{r}) \right\rangle - \bar{E}\bar{E}^*$$

$$= \left\langle e^{i(\Phi(\underline{\xi}) - \Phi(\underline{\xi} + \underline{r}))} \right\rangle - \left\langle e^{i\Phi(\underline{\xi})} \right\rangle \left\langle e^{-i\Phi(\xi)} \right\rangle$$

To obtain the ensemble averages we have to know the probability distribution for the phase fluctuation at any point on the screen. We will assume that the probability distribution is a Gaussian, which is not an unreasonable assumption to make since the phase at any point on the screen is the sum of a large number of small phase fluctuations produced by the medium along the line of sight, and whatever be the probability distribution of the elementary fluctuations, by the Central Limit Theorem, the probability distribution of the sum tends to be a Gaussian. For a Gaussian distribution, the averaging in the above equation can be performed by using a theorem by Mercier (1962) which states that if $\alpha_1, \alpha_2 \ldots \alpha_n$ are n complex quantities whose 2n real and imaginary parts have a Gaussian distribution, then

$$\langle \exp(\alpha_1 + \alpha_2 + \cdots \alpha_n) \rangle = \exp\left[\frac{1}{2}\langle(\alpha_1 + \alpha_2 + \cdots \alpha_n)^2\rangle\right] \quad (2.38)$$

Using this result and the definition of the autocorrelation function of the phase fluctuation

$$\phi_0^2 \rho(\underline{\gamma}) = \langle \phi(\underline{\xi})\,\phi(\underline{\xi} + \underline{\gamma}) \rangle$$

we get

$$f_E(\underline{\gamma}) = \exp -\frac{1}{2}\langle(\phi(\underline{\xi}) - \phi(\underline{\xi} + \underline{\gamma}))^2\rangle - \exp -\phi_0^2$$

$$= e^{-\phi_0^2}\left[e^{\phi_0^2 \rho(\underline{\gamma})} - 1\right] \quad (2.39)$$

The width of $f_E(\underline{\gamma})$ clearly depends on the value of ϕ_0. If $\phi_0 \ll 1$ we have

$$f_E(\underline{\gamma}) \approx \phi_0^2 \rho(\underline{\gamma})$$

and the width of f_E is of the order of a, the scale size of the phase irregularities. For $\phi_o \gg 1$

$$f_E(\underline{\gamma}) \simeq e^{-\phi_o^2(1-\rho(\underline{\gamma}))}$$

Because of the ϕ_o^2 in the exponential, f_E will fall to very small values even for a small decrease of $\rho(\underline{\gamma})$ from 1 and so for this range of $\underline{\gamma}$ we can expand $\rho(\underline{\gamma})$ as a Taylor series and retain only the first two terms, which gives, since the first derivative of $\rho(\underline{\gamma})$ is zero,

$$f_E(\underline{\gamma}) \simeq \exp - \left[\frac{\phi^2}{2} \left. \frac{\partial^2 \rho}{\partial \gamma^2} \right|_{\gamma=0} \gamma^2 \right]$$

where we have assumed that $\rho(\underline{\gamma})$ is circularly symmetrical. We thus see that this approximation f_E behaves as a Gaussian and has a width equal to

$$l = \sqrt{\frac{1}{\phi_o^2 \left. \frac{\partial^2 \rho}{\partial \gamma^2} \right|_{\gamma=0}}} \simeq \frac{a}{\phi_o}$$

Equation (2.37) when written in full is

$$\frac{k\,|\underline{\xi}' - \underline{\xi}|}{\sqrt{z^2 + |\underline{\xi}' - \underline{\xi}|^2_{max}}} = \frac{1}{l}$$

or

$$|\underline{\xi}' - \underline{\xi}|_{max} = \frac{z}{\sqrt{k^2 l^2 - 1}} \simeq \frac{z}{kl} \quad \text{if } kl \gg 1 .$$

Substituting for l we get

$$|\underline{\xi}' - \underline{\xi}|_{max} = z/ka \quad \text{if } \phi_o \ll 1$$

$$= z\phi_o/ka \quad \text{if } \phi_o \gg 1 \qquad (2.40)$$

which is identical to the limit derived from Geometrical optics.

Thus since we are not affected by values of $|\underline{\xi}' - \underline{\xi}|$ larger than given in equation (2.40), the error we make in retaining only the first two terms in the series expansion (2.34) for Υ is roughly equal to the value of the third term at $|\underline{\xi}' - \underline{\xi}|_{max}$ ie. equal to $\frac{|\underline{\xi}' - \underline{\xi}|^4_{max}}{8 z^3}$ and so long as this is much less than a wavelength our approximation is good. Inserting the value $|\underline{\xi}' - \underline{\xi}|_{max}$ the condition for the validity of the approximation is

$$z / 8 k^4 \ell^4 \ll 1$$

$$\text{ie} \quad z\lambda^3 / 128\pi^4 a^4 \ll 1 \qquad \text{if} \quad \phi_0 \ll 1$$

$$z\lambda^3 \phi_0^4 / 128\pi^4 a^4 \ll 1 \qquad \text{if} \quad \phi_0 \gg 1$$

Even if we take the worst possible parameters for the IP medium and put λ = 10 metres, z = 1 A.U., a = 10 km we get $z\lambda^3 / 128\pi^4 a^4 \sim 10^{-5}$ which shows that the approximation is valid for any frequency, for any region of the IP medium provided ϕ_0 is not too large. For λ = 10 m, a = 10 km, ϕ_0 should be less than $\sim$ 10 but for more realistic values of the parameters the approximation is valid for much larger values of ϕ_0.

If we put the expansion of Υ in equation (2.33) we get

$$E(\underline{\xi}) = \frac{k}{2\pi z} e^{ikz} \int e^{i\frac{k}{2z}|\underline{\xi}' - \underline{\xi}|^2} e^{i\phi(\underline{\xi}')} d\underline{\xi}' \qquad (2.41)$$