The intensity at any point in the observers plane is given by

$$I(\underline{\xi}) = E(\underline{\xi})\, E^*(\underline{\xi})$$

$$= \left(\frac{k}{2\pi z}\right)^2 \iint d\underline{\xi}_1\, d\underline{\xi}_2\; e^{i\frac{k}{2z}\left\{(\underline{\xi}_1-\underline{\xi})^2-(\underline{\xi}_2-\underline{\xi})^2\right\}}$$

$$\times\; e^{i\left[\phi(\underline{\xi}_1)-\phi(\underline{\xi}_2)\right]}$$

(2.42)

The autocorrelation function of the intensity fluctuations on the ground is defined as

$$m_z^2\, \rho_I(\underline{\gamma}) = \frac{\left\langle \left[I(\underline{\xi})-\bar{I}\right]\left[I(\underline{\xi}+\underline{\gamma})-\bar{I}\right]\right\rangle}{\bar{I}^{\,2}}$$

$$= \left\langle I(\underline{\xi})\, I(\underline{\xi}+\underline{\gamma})\right\rangle - 1$$

since $\bar{I} = 1$. Substituting for I from (2.42) and taking the angular brackets inside the integrals we get

$$m_z^2\, \rho(\underline{\gamma}) + 1 = \left(\frac{k}{2\pi z}\right)^4 \iiiint d\underline{\xi}_1\, d\underline{\xi}_2\, d\underline{\xi}_3\, d\underline{\xi}_4$$

$$\times \left\langle \exp i\left\{\phi(\underline{\xi}_1)-\phi(\underline{\xi}_2)+\phi(\underline{\xi}_3)-\phi(\underline{\xi}_4)\right\}\right\rangle$$

$$\times \exp\left[i\frac{k}{2z}\left\{(\underline{\xi}_1-\underline{\xi})^2-(\underline{\xi}_2-\underline{\xi})^2+(\underline{\xi}_3-\underline{\xi}-\underline{\gamma})^2-(\underline{\xi}_4-\underline{\xi}-\underline{\gamma})^2\right\}\right]$$

Using equation (2.38) and the definition of the phase autocorrelation function we can write

$$\langle \exp i\{\phi(\underline{\xi}_1) - \phi(\underline{\xi}_2) + \phi(\underline{\xi}_3) - \phi(\underline{\xi}_4)\} \rangle$$

$$= \exp -\frac{1}{2}\langle \{\phi(\underline{\xi}_1) - \phi(\underline{\xi}_2) + \phi(\underline{\xi}_3) - \phi(\underline{\xi}_4)\}^2 \rangle$$

$$= \exp -\phi_0^2 \{2 - \rho(\underline{\xi}_1 - \underline{\xi}_2) + \rho(\underline{\xi}_1 - \underline{\xi}_3) - \rho(\underline{\xi}_1 - \underline{\xi}_4)$$
$$- \rho(\underline{\xi}_2 - \underline{\xi}_3) + \rho(\underline{\xi}_2 - \underline{\xi}_4) - \rho(\underline{\xi}_3 - \underline{\xi}_4)\}$$

$$= F(\underline{\xi}_1, \underline{\xi}_2, \underline{\xi}_3, \underline{\xi}_4)$$

where $F(\underline{\xi}_1, \underline{\xi}_2, \underline{\xi}_3, \underline{\xi}_4)$ is a function of only the difference of the $\underline{\xi}_i$ s taken in pairs. Thus

$$m_z^2 \, \rho_I(\underline{\gamma}) + 1 = \left(\frac{k}{2\pi z}\right)^4 \iiiint d\underline{\xi}_1 \, d\underline{\xi}_2 \, d\underline{\xi}_3 \, d\underline{\xi}_4 \, F(\underline{\xi}_1, \underline{\xi}_2, \underline{\xi}_3, \underline{\xi}_4)$$
$$\times \exp\left[\frac{ik}{2z}\left\{(\underline{\xi}_1 - \underline{\xi})^2 - (\underline{\xi}_2 - \underline{\xi})^2 + (\underline{\xi}_3 - \underline{\xi} - \underline{\gamma})^2 - (\underline{\xi}_4 - \underline{\xi} - \underline{\gamma})^2\right\}\right]$$

(2.43)

The spatial power spectrum of the intensity fluctuations on the ground $M_z^2(\underline{q})$ is the Fourier transform of the acf. Taking the Fourier transform of equation (2.43) we get

$$M_z^2(\underline{q}) + \delta(\underline{q}) = \left(\frac{k}{2\pi z}\right)^4 \iiiint d\underline{\xi}_1 \, d\underline{\xi}_2 \, d\underline{\xi}_3 \, d\underline{\xi}_4 \, F(\underline{\xi}_1, \underline{\xi}_2, \underline{\xi}_3, \underline{\xi}_4)$$
$$\times \int d\underline{\gamma} \; e^{i\underline{q}\cdot\underline{\gamma}} \exp\left[\frac{ik}{2z}\left\{(\underline{\xi}_1 - \underline{\xi})^2 - (\underline{\xi}_2 - \underline{\xi})^2 + (\underline{\xi}_3 - \underline{\xi} - \underline{\gamma})^2 - (\underline{\xi}_4 - \underline{\xi} - \underline{\gamma})^2\right\}\right]$$

Using the fact that $F(\xi_i)$ depends only on the difference on the ξ_i's this five fold integral can be reduced to a single integral (see Appendix A) giving

$$M_z^2(\underline{q}) + \delta(\underline{q}) = \int d\underline{\xi}\; e^{i\underline{q}\cdot\underline{r}}\; F(\underline{\xi}, \underline{\xi} - \tfrac{z}{R}\underline{q}, -\tfrac{z}{R}\underline{q}, 0)$$

If we write F explicitly in terms of the ρ_s we get

$$M_z^2(\underline{q}) = \exp\left[-2\phi_0^2\left(1 - \rho\left(\tfrac{z}{R}\underline{q}\right)\right)\right]$$

$$\times \int d\underline{r}\; e^{i\underline{q}\cdot\underline{r}} \left[\exp \phi_0^2 \left\{2\rho(\underline{r}) - \rho\left(\underline{r} + \tfrac{z}{R}\underline{q}\right) - \rho\left(\underline{r} - \tfrac{z}{R}\underline{q}\right)\right\} - 1\right]$$

(2.44)

This scintillation index m_z is the r.m.s. variation of the intensity on the ground and it is equal to the square root of the intensity acf at zero lag which is related to the power spectrum by

$$m_z^2 = \int d\underline{q}\; M_z^2(\underline{q})$$

Equation (2.44) for the power spectrum cannot be simplified further without making approximations. We will in the following sections derive some expressions for the scintillation index and the power spectrum under various approximations.

2.5 The Weak Scattering Approximation $\phi_0 \ll 1$

When $\phi_0^2 \ll 1$ the exponential in equation (2.44) can be expanded as a series in ϕ_0^2 and only the lowest order terms in ϕ_0^2 need be retained. We then get

$$M_z^2(q) = \phi_0^2 \int d\underline{r}\, e^{i\underline{q}\cdot\underline{r}} \left\{ 2\rho(\underline{r}) - \rho(\underline{r}+\tfrac{z}{k}\underline{q}) - \rho(\underline{r}-\tfrac{z}{k}\underline{q}) \right\} \quad (2.45)$$

Using the definition of the power spectrum of the phase fluctuations

$$\Phi^2(\underline{q}) = \phi_0^2 \int d\underline{r}\, e^{i\underline{q}\cdot\underline{r}}\, \rho(\underline{r})$$

and the shift theorem of Fourier Transforms which says

$$\int d\underline{r}\, e^{i\underline{q}\cdot\underline{r}} f(\underline{r}+\underline{\xi}) = e^{-i\underline{q}\cdot\underline{\xi}} \int d\underline{r}\, e^{i\underline{q}\cdot(\underline{r}+\underline{\xi})} f(\underline{r}+\underline{\xi})$$

$$= e^{-i\underline{q}\cdot\underline{\xi}} \times \text{Fourier transform of } f(\underline{\xi})$$

Equation (2.45) becomes

$$M_z^2(\underline{q}) = \Phi^2(\underline{q}) \left\{ 2 - e^{-i\frac{zq^2}{k}} - e^{i\frac{zq^2}{k}} \right\}$$

$$= 2\Phi^2(\underline{q}) \left\{ 1 - \cos\left(\frac{zq^2}{k}\right) \right\}$$

$$= 4\Phi^2(\underline{q}) \sin^2\left(\frac{zq^2}{2k}\right) \quad (2.46)$$

Thus in the weak scattering approximation the spectrum of the intensity fluctuations on the ground is the same as the spectrum of the phase fluctuations on the screen multiplied by $4\sin^2\left(\frac{zq^2}{2k}\right)$ which will be referred to as the Fresnel filter. If we write

$$q_F = \sqrt{\frac{2k}{z}} = \left(110\ _{km}\right)^{-1} \left(\lambda_{metre}\ z_{A.U.}\right)^{-1/2} \quad (2.47)$$

the Fresnel filter is $4\sin^2\left(q^2/q_F^2\right)$ and all spatial frequencies less than

$\sim q_F$ are heavily attenuated which means that the intensity pattern on the ground is not sensitive to irregularities on the screen with scale sizes larger than about q_F^{-1}. So even if the IP medium contains a hierarchy of scale sizes, by studying IPS we can study the properties of only those irregularities whose size is less than a few hundred kilometres. The reason for this is that the intensity fluctuations arise because of interference between the scattered rays. As we have seen, the field at any point on the ground is affected by only that part of the phase screen that intersects the scattering cone and for intensity fluctuations to occur we must have more than one irregularity in this area. Since the radius of the region of the screen that affects us is $z\theta_{scat} \approx z/ka$ the condition for intensity fluctuations to be observed is $z/ka > a$ or $a < \sqrt{\frac{z}{k}}$ which is equal to $\sim q_F^{-1}$. Irregularities of larger size can produce only phase fluctuations on the ground.

Two quantities that are used to describe the observed scintillation are m_I, the scintillation index and ℓ, the scale size of the intensity fluctuation on the ground. We will use the second moment of the power spectrum of the intensity fluctuations to estimate the scale size. The second moment of the spatial power spectrum in any direction x is defined as

$$q_{2x}^2 = \frac{\int d\underline{q}\; q_x^2\; M_I^2(\underline{q})}{\int d\underline{q}\; M_I^2(\underline{q})} \tag{2.48}$$

and is related to the spatial autocorrelation function by $q_{2x}^2 = -\left[\frac{\partial^2 \rho_I(\underline{r})}{\partial x^2}\right]_{r=0}$. Thus q_{2x} is the reciprocal of the scale size in the x direction. If we assume that the irregularities are circularly symmetrical then $\rho(\underline{r})$

and $M_z^2(\underline{q})$ are functions of $|\underline{y}| + |\underline{q}|$ alone and by writing the integral in polar coordination equation (2.48) simplifies to

$$q_2^2 = \frac{1}{2} \frac{\int_0^\infty dq\, q^3\, M_z^2(q)}{\int_0^\infty dq\, q\, M_z^2(q)} \tag{2.49}$$

where we have dropped the suffix x since q_2 is independent of direction.

With the assumption of circular symmetry the scintillation index is equal to

$$m_z^2 = 2\pi \int_0^\infty dq\, q\, M_z^2(q) \tag{2.50}$$

We will substitute for $M_z^2(\underline{q})$ from equation (2.46) and derive the dependence of m_z and q_2 on ϕ_0 and a. For this we have to assume the form of the autocorrelation function of the phase irregularities. In all our analysis we will unless otherwise specified, assume that the acf has a circularly symmetrical Gaussian structure of the form

$$\rho(\underline{y}) = e^{-y^2/2a^2}$$

so that

$$\bar{\Phi}^2(\underline{q}) = \frac{\phi_0^2 a^2}{2\pi} e^{-\frac{a^2 q^2}{2}} \tag{2.51}$$

and

$$M_z^2(\underline{q}) = \frac{4\phi_0^2 a^2}{2\pi} e^{-\frac{a^2 q^2}{2}} \sin^2\left(\frac{z q^2}{2k}\right) \tag{2.52}$$

Substituting for $M_z^2(q)$ in equation (2.49) and (2.50) we see that to obtain m_z^2 and q_z^2 we have to evaluate two integral of the form

$$I_1(\alpha,\beta) = \int_0^\infty dq\, q\, e^{-\alpha q^2} \sin^2(\beta q^2)$$

$$I_2(\alpha,\beta) = \int_0^\infty dq\, q^3\, e^{-\alpha q^2} \sin^2(\beta q^2)$$

I_1 can be readily evaluated by substituting $y = q^2$ and putting $\sin^2 \beta q^2 = \frac{1}{2}(1-\cos(2\beta q^2))$ giving

$$I_1(\alpha,\beta) = \frac{1}{4}\int_0^\infty dy\, e^{-\alpha y}\,(1-\cos 2\beta y)$$

$$= \frac{1}{4}\int_0^\infty dy \left\{ e^{-\alpha y} - \frac{e^{-(\alpha-2i\beta)y}}{2} - \frac{e^{-(\alpha+2i\beta)y}}{2} \right\}$$

$$= \frac{1}{4}\left[\frac{1}{\alpha} - \frac{1}{2(\alpha-2i\beta)} - \frac{1}{2(\alpha+2i\beta)}\right]$$

$$= \frac{\beta^2}{\alpha(\alpha^2+4\beta^2)} \qquad (2.53)$$

I_2 can be got from I_1 by differentiating it with respect to α.

$$I_2(\alpha,\beta) = -\frac{\partial I_1(\alpha,\beta)}{\partial\alpha} = \frac{\beta^2(3\alpha^2+4\beta^2)}{\alpha^2(\alpha^2+4\beta^2)^2} \tag{2.54}$$

The scintillation index is given by

$$m_z^2 = 2\pi\int_0^\infty dq\, q\, M_z^2(q) = 4\phi_0^2 a^2\, I_1\left(\frac{a^2}{2},\frac{z}{2k}\right)$$
$$= 2\phi_0^2 \Big/ \left[1+(z_0/2z)^2\right] \tag{2.55}$$

where $z_0 = ka^2$ is called the Fresnel distance.

The second moment of the intensity spectrum is given by

$$q_{1/2}^2 = \frac{1}{2}\,\frac{I_2\left(\frac{a^2}{2},\frac{z}{2k}\right)}{I_1\left(\frac{a^2}{2},\frac{z}{2k}\right)}$$
$$= \frac{1}{a^2}\left[1+2\Big/\left\{1+(2z/z_0)^2\right\}\right] \tag{2.56}$$

The behaviour of the scintillation under two limiting cases is of interest.

In the far field approximation where $z/z_0 \gg 1$ the $\sin^2$ term in equation (2.46) oscillates very rapidly and can be replaced by its mean value of 1/2 giving

$$M_z^2(q) = 2\bar{\Phi}^2(q)$$

$$m_z^2 = 2\phi_0^2 \tag{2.57}$$

$$q_2^2 = \frac{1}{a^2} \tag{2.58}$$

In this limit the scale size of the intensity fluctuations on the ground is the same as the scale size on the screen and the scintillation index has its largest value for a given value of ϕ_0. As z decreases the scintillation index and the scale size on the ground decrease because low frequencies in the power spectrum get filtered out by the Fresnel filter. When $z/z_0 \ll 1$ we have

$$m_z^2 = 8\left(\frac{z\phi_0}{z_0}\right)^2$$
$$q_2^2 = 3/a^2 \tag{2.59a}$$

By putting sin θ = θ in equation (2.46) we get the power spectrum in this approximation as

$$M_z^2(q) = \left(q^2 z/k\right)^2 \bar{\Phi}^2(q) \tag{2.59b}$$

In figure 3 we have shown the variation of m_z and q_2 with a distance parameter

$$d = \frac{2z}{z_0} = \frac{z\lambda}{\pi a^2} = 5\,\frac{z_{A.U.}\,\lambda_{metre}}{a^2_{100\,km}} \tag{2.60}$$

If we assume that $z >$ 0.7 A.U., which is true for our observations, and take a = 100 km, then at 327 MHz $d > 3$ for all our

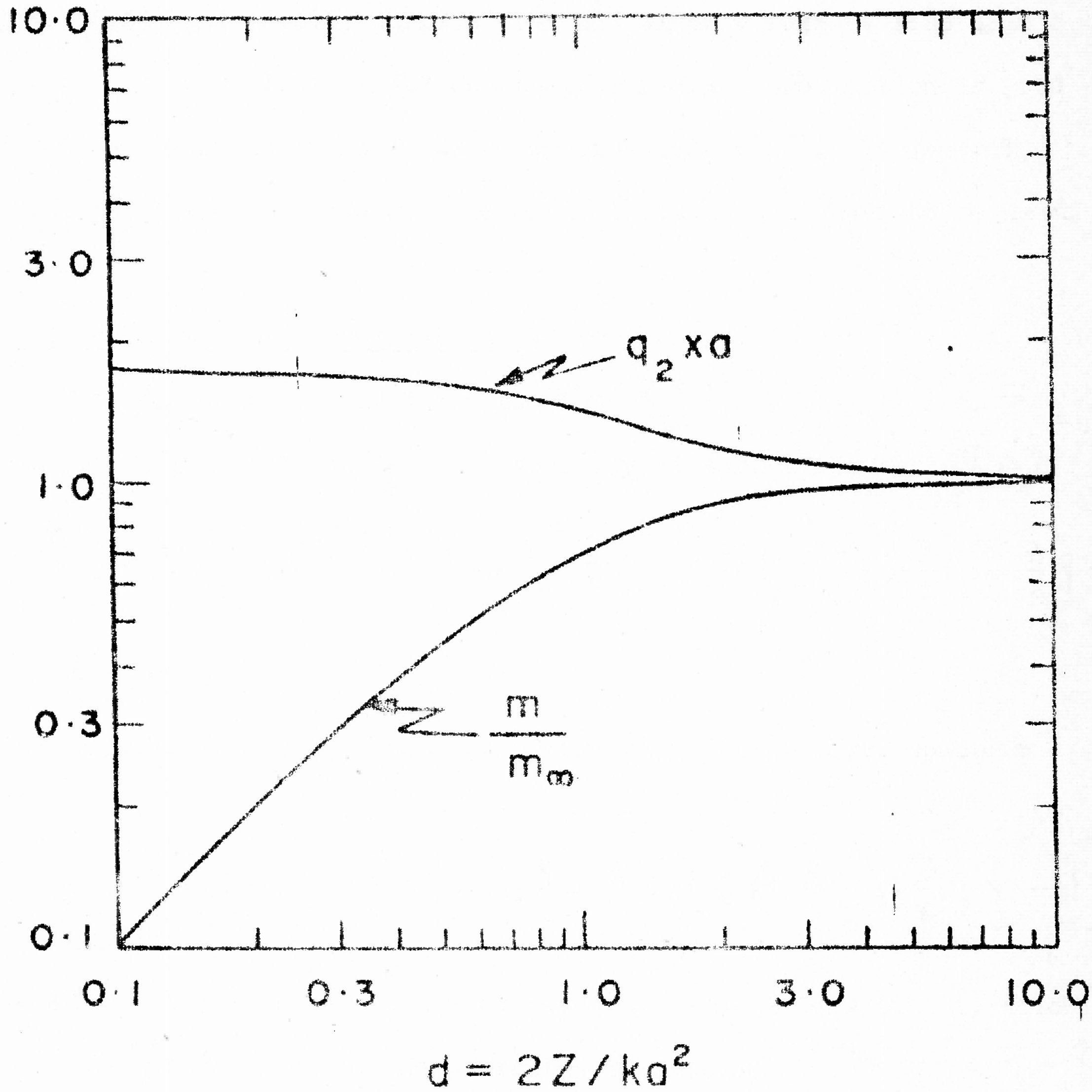

Figure 3. Dependence of the scintillation index and second moment of the spatial power spectrum on distance from the screen when the scattering is weak ($\phi_0^2 \ll 1$).

observations. From figure 3, the far field approximation for m and q_2 (equations (2.57) and (2.58)) differs from the exact values by less than 5 and 10% respectively. Thus at 327 MHz the far field approximation is good for the range of elongations over which observations exist. In general, however, this will not be true and the exact formulae will have to be used.

2.5 Geometrical optics Approximation $\lambda \to 0$ or $k \to \infty$

If k is sufficiently large so that $\frac{z q_{max}}{k} \sim \frac{z}{ka} \ll a$ i.e. $z/z_0 \ll 1$ we can expand $\rho(\underline{r} \pm \frac{z}{k}\underline{q})$ as a Taylor series about $\rho(\underline{r})$

$$\rho(\underline{r} \pm \tfrac{z}{k}\underline{q}) = \rho(\underline{r}) \pm \nabla\rho(\underline{r}) \cdot \tfrac{z}{k}\underline{q} + \tfrac{1}{2}\nabla\left(\nabla\rho(\underline{r}) \cdot \tfrac{z}{k}\underline{q}\right) \cdot \tfrac{z}{k}\underline{q} + \cdots$$

Also

$$\rho(\tfrac{z}{k}\underline{q}) \approx 1$$

With these approximations equation (2.44) for the spatial power spectrum reduces to

$$M_z^2(\underline{q}) = \int d\underline{r}\, e^{i\underline{q}\cdot\underline{r}} \left[\exp\left\{-\phi_0^2 \nabla\left(\nabla\rho \cdot \tfrac{z}{k}\underline{q}\right) \cdot \tfrac{z}{k}\underline{q}\right\} - 1\right]$$

If, in addition, the argument of the exponential is much less than 1, which requires (using order of magnitude estimates by putting $\nabla\rho \sim \frac{1}{a}$ & $\underline{q} \sim q_{max} \sim \frac{1}{a}$)

$$\phi_0^2 z^2/z_0^2 \ll 1 \qquad \text{ie} \qquad \phi_0 z/z_0 \ll 1$$

we get

$$M_z^2(\underline{q}) = -\phi_0^2 \int d\underline{r}\, e^{i\underline{q}\cdot\underline{r}}\, \nabla\left(\nabla\rho \cdot \tfrac{z}{k}\underline{q}\right) \cdot \tfrac{z}{k}\underline{q} \qquad (2.61)$$

This can be integrated fairly simply if we write the integral in cartesian coordinates. We have

$$\nabla\rho\cdot\frac{Z}{k}\underline{q} = \frac{Z}{k}\left[\frac{\partial\rho}{\partial x}q_x + \frac{\partial\rho}{\partial y}q_y\right]$$

$$\nabla\left(\nabla\rho\cdot\frac{Z}{k}\underline{q}\right)\cdot\frac{Z}{k}\underline{q} = \left(\frac{Z}{k}\right)^2\left[\frac{\partial^2\rho}{\partial x^2}q_x^2 + 2\frac{\partial^2\rho}{\partial x\partial y}q_xq_y + \frac{\partial^2\rho}{\partial y^2}q_y^2\right]$$

so

$$M_Z^2(q_x,q_y) = -\phi_0^2\left(\frac{Z}{k}\right)^2\iint dx\,dy\; e^{i(xq_x+yq_y)} \times \left[\frac{\partial^2\rho}{\partial x^2}q_x^2 + 2\frac{\partial^2\rho}{\partial x\partial y}q_xq_y + \frac{\partial^2\rho}{\partial y^2}q_y^2\right]$$

Taking one term

$$\iint_{-\infty}^{\infty} dx\,dy\; e^{i(xq_x+yq_y)}\frac{\partial^2\rho}{\partial x^2}q_x^2 = q_x^2\int_{-\infty}^{\infty}dy\,e^{iyq_y} \times \int_{-\infty}^{\infty}dx\,e^{ixq_x}\frac{\partial^2\rho}{\partial x^2}$$

$$= -q_x^4\iint_{-\infty}^{\infty}dx\,dy\;\rho(x,y)\,e^{i(xq_x+yq_y)}$$

where we have performed the integration over x by parts and assumed that ρ and $\frac{\partial\rho}{\partial x}$ vanish at $\pm\infty$. The other two terms can be similarly calculated. If we put these results back in equation (2.61) we get

$$M_Z^2(q_x,q_y) = \phi_0^2\left(\frac{Z}{k}\right)^2\left(q_x^4 + 2q_x^2q_y^2 + q_y^4\right)\iint dx\,dy\;e^{i\underline{q}\cdot\underline{r}}\rho(\underline{r})$$

$$= \left(\phi_0 Z/k\right)^2\left(q_x^2+q_y^2\right)^2\int d\underline{r}\;e^{i\underline{q}\cdot\underline{r}}\rho(\underline{r})$$

$$= \frac{Z^2q^4}{k^2}\,\Phi^2(\underline{q}) \tag{2.62}$$

This expression is identical to equation (2.59b) which was derived assuming $\phi_o \ll 1$ and $z/z_o \ll 1$. The present result was derived under the less restrictive assumptions that. $z/z_o \ll 1$ and $z\phi_o/z_o \ll 1$, from which it is clear that this expression is valid even for $\phi_o > 1$.

The Geometrical optics approximation is valid if the phase variations are small over that portion of the phase screen that intersects the scattering cone because then, wave effects like interference between wavelets from different parts of the phase screen can be safely neglected. In the weak scattering region the condition for this was derived as $z/z_o \ll 1$. In the strong scattering case if we remember that the angle of scattering is ϕ_o/ka, the radius of the screen affecting the field is $z\phi_o/ka$. Setting this less than the scale size we immediately get the second condition $z\phi_o/z_o \ll 1$ also. The expressions for the scintillation index and the second moment are the same as given in equation (2.59). They are

$$m_z^2 = 8\left(\frac{z\phi_o}{z_o}\right)^2$$

$$q_{v2}^2 = 3/a^2$$

2.7 Far Field Approximation

Let us assume that the phase autocorrelation function has a cut off at $\gamma = A$ i.e. $\rho(\underline{\gamma}) = 0$ for $|\underline{\gamma}| \geq A$. Let us define two functions and their Fourier transforms

$$g(\underline{\gamma}) = \left(e^{2\phi_o^2 \rho(\underline{\gamma})} - 1\right), \quad g(\underline{\gamma}) \overset{F.T.}{\rightleftharpoons} G_1(\underline{q})$$

$$f(\underline{\gamma}) = \left(e^{-\phi_o^2 \rho(\underline{\gamma})} - 1\right), \quad f(\underline{\gamma}) \overset{F.T.}{\rightleftharpoons} F(\underline{q})$$

Both $g(\underline{r})$ and $f(\underline{r})$ are equal to 0 for $|\underline{r}| \geq A$. With these definitions, equation (2.44) for the power spectrum becomes

$$M_z^2(\underline{q}) = \exp\left[-2\phi_0^2\left\{1 - \rho\left(\frac{z}{R}\underline{q}\right)\right\}\right] \times \int d\underline{r}\, e^{i\underline{q}\cdot\underline{r}}$$

$$\times\left[\left(g(\underline{r})+1\right)\left(f\left(\underline{r}+\frac{z}{R}\underline{q}\right)+1\right)\left(f\left(\underline{r}-\frac{z}{R}\underline{q}\right)+1\right) - 1\right]$$

$$= \exp\left[-2\phi_0^2\left(1-\rho\left(\frac{z}{R}\underline{q}\right)\right)\right] \times \int d\underline{r}\; e^{i\underline{q}\cdot\underline{r}} \times$$

$$\Big[g(\underline{r}) + f\left(\underline{r}+\frac{z}{R}\underline{q}\right) + f\left(\underline{r}-\frac{z}{R}\underline{q}\right) + g(\underline{r})f\left(\underline{r}+\frac{z}{R}\underline{q}\right) +$$

$$g(\underline{r})f\left(\underline{r}-\frac{z}{R}\underline{q}\right) + f\left(\underline{r}+\frac{z}{R}\underline{q}\right)f\left(\underline{r}-\frac{z}{R}\underline{q}\right) + g(\underline{r})f\left(\underline{r}+\frac{z}{R}\underline{q}\right)f\left(\underline{r}-\frac{z}{R}\underline{q}\right)\Big]$$

Let us assume that z is sufficiently large so that for any $\underline{q}$ we choose, $\left|\frac{z}{R}\underline{q}\right| > A$. Then the 3 functions $g(\underline{r})$, $f\left(\underline{r}+\frac{z}{R}\underline{q}\right)$ and $f\left(\underline{r}-\frac{z}{R}\underline{q}\right)$ will be non zero in non-overlapping regions of $\underline{r}$ and so the various products of these terms are equal to zero for all $\underline{r}$. Under this condition

$$M_z^2(\underline{q}) = e^{-2\phi_0^2}\int d\underline{r}\left[g(\underline{r}) + f\left(\underline{r}+\frac{z}{R}\underline{q}\right) + f\left(\underline{r}-\frac{z}{R}\underline{q}\right)\right]e^{i\underline{q}\cdot\underline{r}}$$

$$= e^{-2\phi_0^2}\left[G(\underline{q}) + e^{-i\frac{z}{R}q^2}F(\underline{q}) + e^{i\frac{z}{R}q^2}F(\underline{q})\right]$$

$$= e^{-2\phi_0^2}G(\underline{q}) + 2e^{-2\phi_0^2}F(\underline{q})\cos\left(\frac{z}{R}q^2\right)$$

As z becomes larger and larger the lower limit on q for which this result is valid becomes smaller and smaller and in the limit $z \to \infty$ this is valid for all $\underline{q}$. In the limit when $z \to \infty$ the second term oscillates about zero with infinite frequency. If we consider the fact that any measurement of the spectrum has a finite frequency resolution and so what one measures is the average value of $M_z^2(\underline{q})$ over a small frequency range, then it is clear that the second term in equation (2.63) can be ignored since its average value over any frequency range is zero. Thus, as $z \to \infty$

$$M_z^2(\underline{q}) = e^{-2\phi_0^2}\, G(\underline{q})$$

The inverse Fourier transform can be readily performed and we get the intensity autocorrelation function as

$$m_z^2\, \rho_I(\underline{r}) = e^{-2\phi_0^2}\, g(\underline{r}) = e^{-2\phi_0^2}\left[e^{2\phi_0^2 \rho(\underline{r})} - 1\right] \tag{2.64}$$

This result (which is valid for all ϕ_0) was first derived by Mercier (1962), though using a different approach.

The scintillation index is given by

$$m_z^2 = e^{-2\phi_0^2}\left(e^{2\phi_0^2} - 1\right) = 1 - e^{-2\phi_0^2} \tag{2.65}$$

The second moment q_2 of the power spectrum of the intensity fluctuations is given by (assuming $\rho(\underline{r})$ is circularly symmetrical),

$$q_2^2 = \frac{1}{2}\, \frac{\int_0^\infty M_z^2(q)\, q^3\, dq}{\int_0^\infty M_z^2(q)\, q\, dq} = \left[-\frac{\partial^2 \rho_I(\underline{r})}{\partial r^2}\right]_{r=0}$$

From equation (2.64)

$$\rho_I(\gamma) = \frac{e^{2\phi_o^2 \rho(\gamma)} - 1}{e^{2\phi_o^2} - 1}$$

so that

$$\frac{\partial^2 \rho_I(\gamma)}{\partial \gamma^2} = \frac{e^{2\phi_o^2 \rho(\gamma)}}{e^{2\phi_o^2} - 1} \left\{ 2\phi_o^2 \frac{\partial^2 \rho}{\partial \gamma^2} + 4\phi_o^4 \left(\frac{\partial \rho}{\partial \gamma}\right)^2 \right\}$$

Since $\left.\frac{\partial \rho}{\partial \gamma}\right|_{\gamma=0} = 0$ for an autocorrelation function,

$$q_2^2 = \frac{2\phi_o^2 e^{2\phi_o^2}}{e^{2\phi_o^2} - 1} \left.\frac{\partial^2 \rho}{\partial \gamma^2}\right|_{\gamma=0} = \frac{2\phi_o^2}{a^2\left(1 - e^{-2\phi_o^2}\right)} \tag{2.66}$$

For $\phi_o^2 \ll 1$ these two expressions reduce to $m_z^2 = 2\phi_o^2$ and $q_2^2 = 1/a^2$ which is similar to equations (2.57, 58). For large values of ϕ_o however

$$m_z^2 = 1$$
$$q_2^2 = 2\phi_o^2 / a^2 \tag{2.67}$$

So we see that for small values of ϕ_o the scale size on the ground is the same as the scale size on the screen and the scintillation index increases linearly with ϕ_o. As ϕ_o becomes of the order of one the scale size on the ground starts reducing and the rate at which m_z increases with ϕ_o drops. For large values of ϕ_o, m_z saturates at 1 and the scale size decreases as ϕ_o^{-1}. The complete dependence of m_z and q_2 on ϕ_o is shown in figure 4.

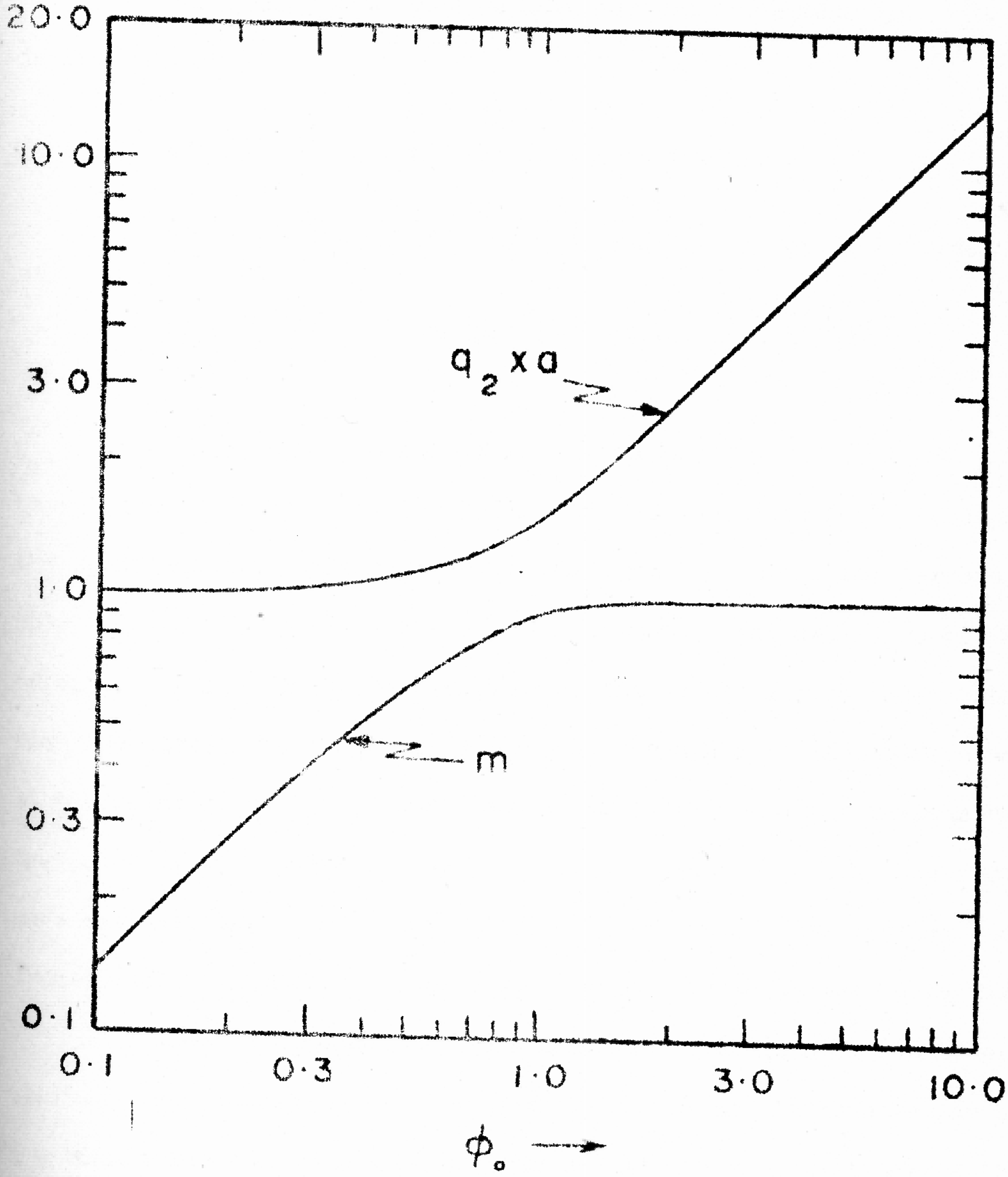

Figure 4. Dependence of the scintillation index and second moment of the spatial power spectrum on ϕ_o in the far field region ($Z/Z_o \gg 1$).

2.8 Numerical Methods

Using approximations we have derived expression for m_z and q_z in the regions

a) $\phi_o \ll 1$, all z (Weak scattering)

b) $z/z_o \ll 1$, $\phi_o\, z/z_o \ll 1$ (near screen or Geometrical optics)

c) $z/z_o \rightarrow \infty$ all ϕ_o (Far field approximation)

To get any information in the region $z/z_o \sim 1$, $\phi_o \gtrsim 1$ we have to use numerical methods. Mercier (1962) studied the dependence of the scintillation index on the distance from the screen for various ϕ_o upto $\phi_o^2 = 2$ and pointed out for the first time the existence of focussing effects. If one plots the variation of the scintillation index with distance from the screen, for $\phi_o^2 > 1$, m_z goes through a maximum before saturating at the value given by the far field approximation and the maximum value of the scintillation index can be much larger than 1. Bramley and Young (1967) have extended the calculation to $\phi_o^2 = 10$ and have found similar results. Bramley and Young have also computed the dependence of the scale size on ϕ_o.

The basic approach in the numerical methods has been to express the required quantity as a power series in ϕ_o and to numerically evaluate the coefficients of this series. For illustration we will derive the series expansion for the scintillation index. Equation (2.44) gives the power spectrum of the intensity fluctuations as

$$M_z^2(\underline{q}) = e^{-2\phi_0^2(1-\rho(\frac{z}{k}\underline{q}))} \int d\underline{r}\; e^{i\underline{q}\cdot\underline{r}}$$

$$\left[\exp \phi_0^2\left\{2\rho(\underline{r}) - \rho(\underline{r}+\tfrac{z}{k}\underline{q}) - \rho(\underline{r}-\tfrac{z}{k}\underline{q})\right\} - 1\right]$$

The scintillation index is given by

$$m_z^2 = \int d\underline{q}\; M_z^2(\underline{q}) = \iint d\underline{r}\, d\underline{q}\; e^{-2\phi_0^2(1-\rho(\frac{z}{k}\underline{q}))}$$

$$e^{i\underline{q}\cdot\underline{r}}\left[\exp \phi_0^2\left\{2\rho(\underline{r}) - \rho(\underline{r}+\tfrac{z}{k}\underline{q}) - \rho(\underline{r}-\tfrac{z}{k}\underline{q})\right\} - 1\right]$$

Using the series expansion for the exponential and taking the summation sign outside the integrals we get

$$m_z^2 = e^{-2\phi_0^2} \sum_{n=1}^{\infty} \frac{\phi_0^{2n}}{n!} \iint d\underline{r}\, d\underline{q}\; e^{i\underline{q}\cdot\underline{r}}$$

$$\left\{2\rho(\underline{r}) + 2\rho(\tfrac{z}{k}\underline{q}) - \rho(\underline{r}+\tfrac{z}{k}\underline{q}) - \rho(\underline{r}-\tfrac{z}{k}\underline{q})\right\}^n$$

Using the expansion

$$(2a+2b-c-d)^n = n! \sum_k \frac{(-)^{k_3+k_4}\, 2^{k_1+k_2}}{k_1!\,k_2!\,k_3!\,k_4!}\, a^{k_1} b^{k_2} c^{k_3} d^{k_4}$$

where the summation includes all combinations of positive and zero values of k_1, k_2, k_3 and k_4 subject to the constraint $k_1 + k_2 + k_3 + k_4 = n$, we get

$$m_z^2 = e^{-2\phi_0^2} \sum_{n=1}^{\infty} \phi_0^{2n} \sum_{k} \frac{(-)^{k_3+k_4}\, 2^{k_1+k_2}}{k_1!\, k_2!\, k_3!\, k_4!}\, I(k_1, k_2, k_3, k_4, z/k)$$

where

$$I(k_1, k_2, k_3, k_4, z/k) = \iint d\underline{q}\, d\underline{r}\; e^{i\underline{q}\cdot\underline{r}}\; \rho^{k_1}(\underline{r})\, \rho^{k_2}\!\left(\frac{z}{k}\underline{q}\right) \times$$

$$\rho^{k_3}\!\left(\underline{r} - \frac{z}{k}\underline{q}\right)\, \rho^{k_4}\!\left(\underline{r} + \frac{z}{k}\underline{q}\right)$$

To evaluate this integral one has to assume the functional form for $\rho(\underline{r})$. If we take ρ to be a circularly symmetrical Gaussian of the form

$$\rho(\underline{r}) = e^{-r^2/2a^2}$$

and write everything in cartesian coordinates, we see that the integrand can be written as a product of two identical functions, one involving only the x component of $\underline{r}$ and $\underline{q}$, and the other involving only the y components. So the integral can be written as

$$I(k_1, k_2, k_3, k_4, z/k) = J^2(k_1, k_2, k_3, k_4, z/k)$$

where

$$J(k_1, k_2, k_3, k_4, z/k) = \iint_{-\infty}^{\infty} dx\, dq_x\, e^{i q_x x} \times$$

$$\exp -\frac{1}{2a^2}\left\{ k_1 x^2 + k_2 \frac{z^2 q_x^2}{k^2} + k_3\left(x - \frac{z}{k}q_x\right)^2 + k_4\left(x + \frac{z}{k}q_x\right)^2 \right\}$$

The evaluation of this integral is straight forward but tedious. One first makes the terms containing x a perfect square, integrates over x, then makes the terms containing q_x a perfect square and integrates over q_x, to get

$$J(k_1, k_2, k_3, k_4, z/k) = \left[1 + \frac{d^2}{4}\left\{(n-k_1)(n-k_2) - (k_3-k_4)^2\right\} + i d (k_4 - k_3)\right]$$

where $d = 2z/z_0$, as defined earlier. The scintillation index is then given by

$$m_z^2 = e^{-2\phi_0^2} \sum_{n=1}^{\infty} \phi_0^{2n} \sum_{k_i} \frac{(-)^{k_3+k_4}\, 2^{k_1+k_2}}{k_1!\, k_2!\, k_3!\, k_4!} \times$$

$$\left[1 + \frac{d^2}{4}\left\{(n-k_1)(n-k_2) - (k_4-k_3)^2\right\} + i d (k_4 - k_3)\right]$$

This expression is identical to the expression derived by Bramley and Young (1967), if one takes into account the difference in the definition of the acf (they define $\rho(\underline{r}) = e^{-r^2/r_0^2}$). It is this series that is summed on a computer. The largest value on n upto which the summation is carried out is determined by the largest value of ϕ_0 one is interested in. Bramley and Young have gone upto $n = 40$ to get the scintillation index for values of ϕ_0^2 upto 10. In figure 5 we have reproduced their results on the variation of m_z^2 with d for various values of ϕ_0^2. The focussing effect is clearly seen. It is also seen that the maximum occurs at smaller values of d as we go to larger values of ϕ_0, and Bramley and Young find that d_{max} and ϕ_0 are related by

$$d_{max}\, \phi_0^{1.2} = 1.35$$

Bramley and Young have also computed the variation of the scale size on the ground with d and ϕ_0. In figure 6 we have reproduced their results giving the variation of u_0, which is the distance measured in units of the scale size at which the acf falls to e^{-1}. It is difficult to get a simple relation between u_0 and the scale size as we have defined it ($-\left.\frac{\partial^2 \rho}{\partial r^2}\right|_{r=0}$), but we will assume that the two scale sizes are approximately proportional to each other.

In figures 7 and 8 we have shown the variation of numerically estimated values of m_z and ℓ with ϕ_0 for different values of d. On the same figures we have, for comparison, plotted the prediction of the weak scattering and the Far field approximation.

2.9 Effects of Source Structure

So far we have considered a plane wave to be incident on the phase screen i.e. we have considered the source of radiation to have no angular diameter. Since one of the main aims of IPS is to get an idea of the angular diameter of the scintillating source, we have to find out how the structure of the source affects the observed quantities. Let $p(\underline{\theta}) = p(\theta_x, \theta_y)$ normalised so that $\int p(\underline{\theta})\, d\underline{\theta} = 1$, be the brightness distribution of the source. The effect of the finite source size is to reduce the coherence of the radiation incident on the phase screen. Now, instead of a single plane wave incident on the screen we have incident a number of plane waves, each travelling in a different direction — what is called the angular spectrum. Because of this if we take a plane perpendicular to the line of sight to the source and measure the acf of the electric field, the acf will not be constant at 1

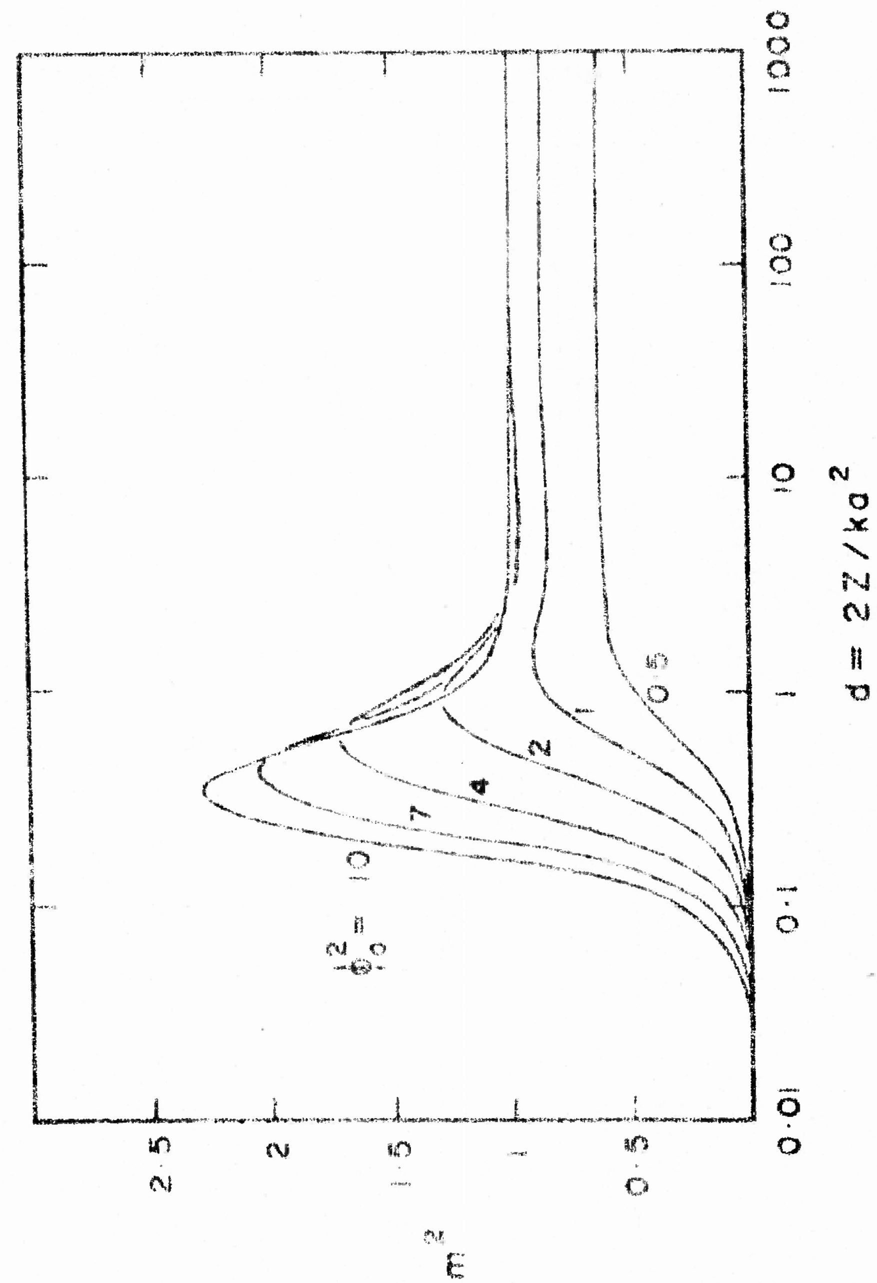

Figure 5. Dependence of the scintillation index on distance from the screen, for different values of ϕ_0. Taken from the numerical computations of Bramley and Young (1967).

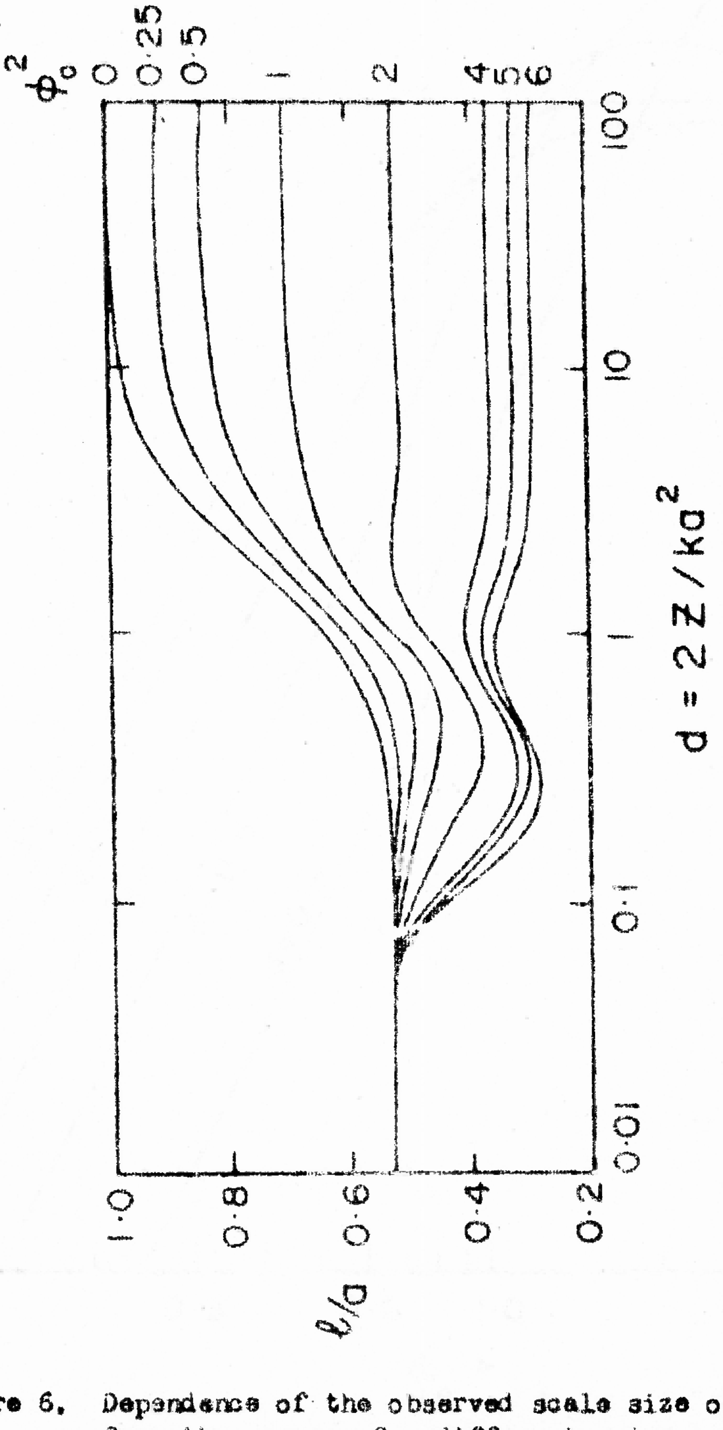

Figure 6. Dependence of the observed scale size on distance from the screen, for different values of ϕ_o. Taken from the numerical computations of Bramley and Young (1967).

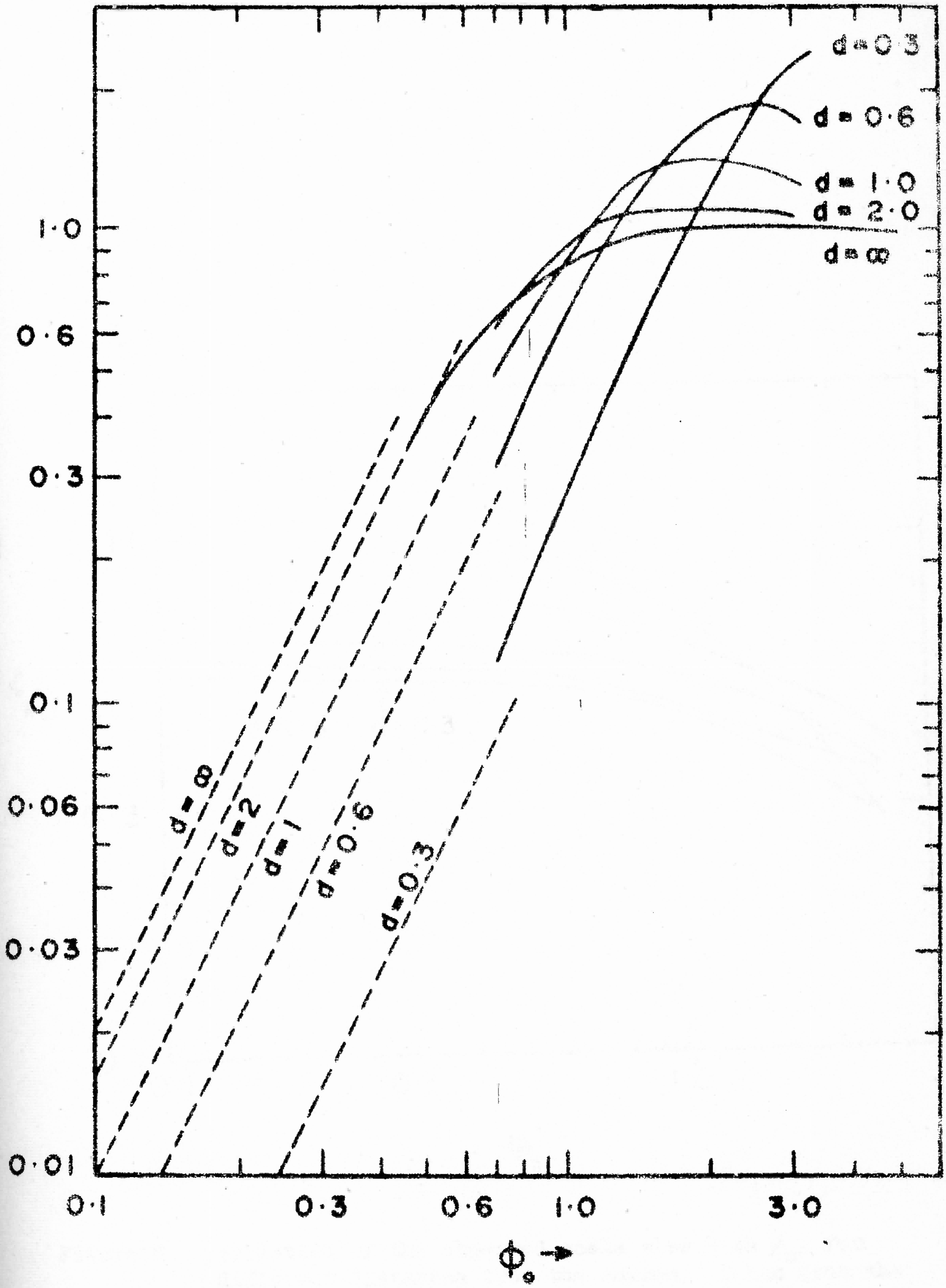

Figure 7. Variation of scintillation index with ϕ_0 for different distances from the screen. The solid lines are from the computations of Bramley and Young (1967) and the dashed lines give the scintillation index in the weak scattering regime.

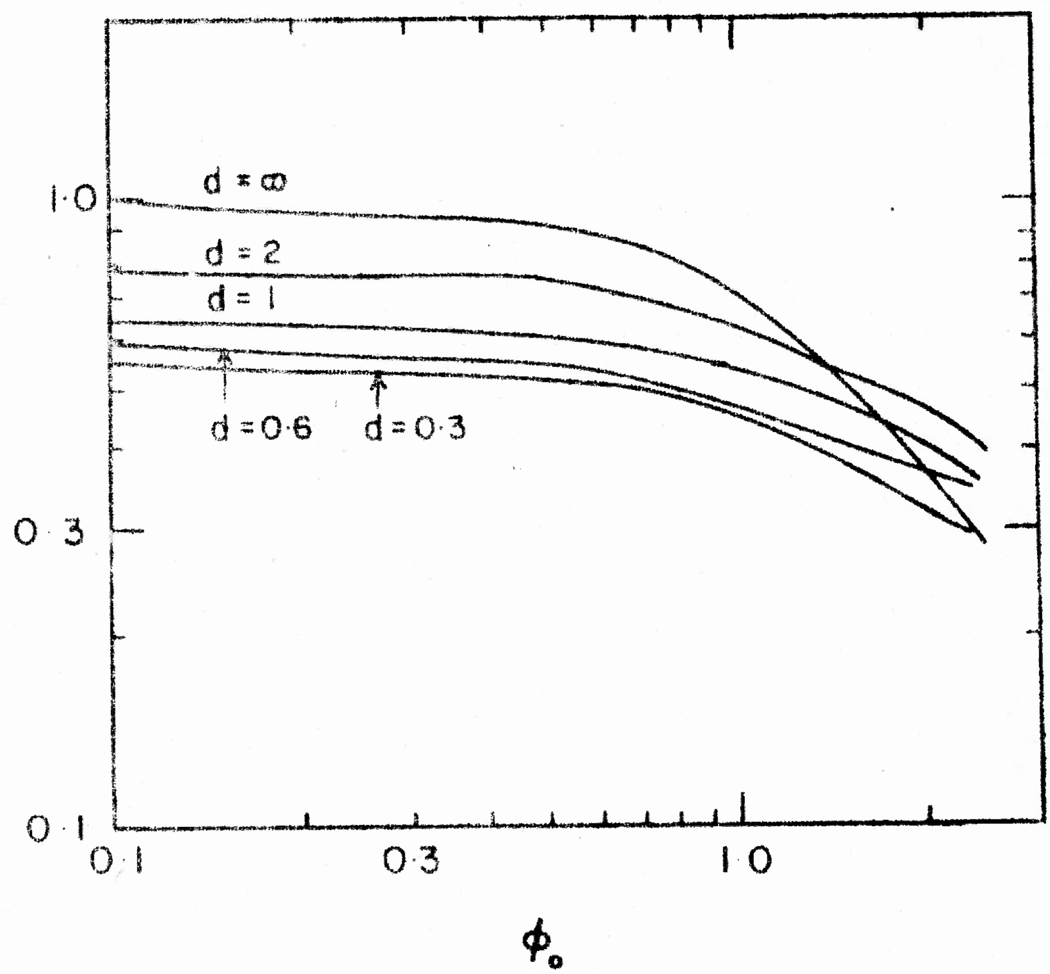

Figure 8. Variation of the observed scale size with ϕ_o, for different distances from the screen. Taken from the computation of Bramley and Young (1967).

for all separations, as is the case for a plane wave, but will decrease as we go to larger separations. If the separation is measured in units of λ, this acf is the same as the visibility function $V(u, v)$ used in radio interferometry.

$$V(\underline{\omega}) = V(u, v) = \langle E(\underline{\xi})\, E^*(\underline{\xi} + \lambda \underline{\omega}) \rangle$$

It is well known (Kraus, 1966) that the visibility function is the same as the Fourier transform of the brightness distribution of the source, i.e.

$$V(\underline{\omega}) = \int d\underline{\theta}\; e^{2\pi i\, \underline{\omega}\cdot\underline{\theta}}\; p(\underline{\theta}) \tag{2.67}$$

Because it is the Fourier transform of the brightness distribution of the source, the width of the visibility function is given by $1/2\pi\,\theta_{source}$ where θ_{source} is the diameter of the source. Thus the actual width of the acf of the electric field is roughly $1/k\theta_{source}$. If the width of the acf is much larger than the region of the phase screen that affects the field at any point on the ground, which is given by $z\theta_{scat}$ the intensity pattern on the ground will not differ much from that of a point source. So the condition for the source size to affect the intensity pattern on the ground is

$$z\theta_{scat} > 1/k\theta_{source}$$

which becomes

$$z\theta_{source} > a \qquad \text{if} \quad \phi_0 << 1$$

$$z\theta_{source} > a/\phi_0 \qquad \text{if} \quad \phi_0 >> 1$$

Another way of looking at the problem is as follows. If we move a point source through a small angle $\Delta\theta$, the effect on the intensity pattern on the ground is to shift it in the opposite direction through a linear distance $z\Delta\theta$. If the angular diameter of the source is so large that $z\Delta\theta$ is greater than the scale size of the intensity fluctuations on the ground then the intensity pattern on the ground gets smeared out and the scintillation index is reduced. Thus the condition for the source size to affect the scintillation is

$$z\theta_{source} > \ell$$

which is

$$z\theta_{source} > a \qquad \text{if} \quad \phi_0 << 1$$

$$z\theta_{source} > a/\phi_0 \qquad \text{if} \quad \phi_0 >> 1$$

which is the same as what we derived earlier. One can go further. If $I(\underline{\xi})$ is the intensity pattern produced by a point source at $\underline{\theta} = 0$, a point source at $\underline{\theta}$ produces an intensity pattern $I(\underline{\xi} - z\underline{\theta})$. If the source has a brightness distribution $p(\theta)$, the intensity pattern on the ground is

$$I_{ext}(\underline{\xi}) = \int d\underline{\theta}\; p(\underline{\theta})\, I(\underline{\xi} - z\underline{\theta}) = \frac{1}{z^2}\int d\underline{\xi}'\, p(\underline{\xi}'/z)\, I(\underline{\xi}-\underline{\xi}')$$

$$= \frac{1}{z^2}\, p(\underline{\xi}/z) * I(\underline{\xi})$$

where we have used the symbol * for convolution. Using the standard results of Fourier transforms (Bracewell, 1965) that if

$$f(\underline{x}) \overset{F.T.}{\rightleftharpoons} F(\underline{k}) \quad \& \quad g(\underline{x}) \rightleftharpoons G(\underline{k})$$

then $f(\underline{x}) * g(\underline{x}) \overset{F.T.}{\rightleftharpoons} F(\underline{k})\,G(\underline{k})$ Convolution theorem

$f(\underline{x}/a) \overset{F.T.}{\rightleftharpoons} a^2 F(a\underline{k})$ Scaling theorem we get for the power spectrum of the intensity fluctuations

$$M_z^2(\underline{q}) = |F.T.\ I_{z,t}(\underline{\xi})|^2 = \left|V\left(\frac{z\underline{q}}{\frac{k}{2\pi}}\right)\right|^2 M_{pt\ source}(\underline{q}) \qquad (2.68)$$

where V is the visibility function of the source. The extra 2π comes in the visibility function because of the difference between our definition of the Fourier transforms and the definition in equation (2.67).

This result can also be derived using the methods we have developed earlier. Because of the finite source size, the phase of the incident radiation is not constant in the $z = 0$ plane but varies with position. The relative phase between two points fluctuates in time with a time scale $\sim 1/\Delta\nu$ where $\Delta\nu$ is the bandwidth of the radiation. The field of the radiation can be characterised by its acf, the visibility function given by

$$V(\underline{u}) = \langle E_0(\underline{\xi})\, E_0^*(\underline{\xi} + \lambda\underline{u}) \rangle$$

The electric field at the observer's plane is got by generalizing equation (2.41) to give

$$E(\underline{\xi}) = \frac{k}{2\pi z} e^{ikz} \int d\underline{\xi}'\, E_0(\underline{\xi}')\, e^{i\phi(\underline{\xi}')}\, e^{i\frac{k}{2z}|\underline{\xi}' - \underline{\xi}|^2}$$

The intensity is given by

$$I(\underline{\xi}) = \left(\frac{k}{2\pi z}\right)^2 \iint d\underline{\xi}_1\, d\underline{\xi}_2\, E_0(\underline{\xi}_1)\, E^*(\underline{\xi}_2)\, e^{i(\phi(\underline{\xi}_1) - \phi(\underline{\xi}_2))}$$

$$e^{i\frac{k}{2z}\left\{(\underline{\xi}_1 - \underline{\xi})^2 - (\underline{\xi}_2 - \underline{\xi})^2\right\}}$$

Since any measurement of I is made with instruments which have a finite time constant and since the time constant is generally much larger than $1/\Delta\nu$, each measured value of I is the average over a large number of fluctuations of the phase of $E_0(\underline{\xi}_\rho)$. So we will replace I by its average value which is got by replacing $E_0(\underline{\xi}_1)\,E_0^*(\underline{\xi}_2)$ by its average value which by definition is $V\left(\frac{\underline{\xi}_1-\underline{\xi}_2}{\lambda}\right)$. With this the expression for the intensity becomes

$$I(\underline{\xi}_\rho) = \left(\frac{k}{2\pi z}\right)^2 \iint d\underline{\xi}_1\, d\underline{\xi}_2\, V\left(\frac{\underline{\xi}_1-\underline{\xi}_2}{\lambda}\right) e^{i\{\phi(\underline{\xi}_1)-\phi(\underline{\xi}_2)\}}\, e^{i\frac{k}{2z}\{(\underline{\xi}_1-\underline{\xi}_\rho)^2-(\underline{\xi}_2-\underline{\xi}_\rho)^2\}}$$

As before we can define the spatial acf of the intensity fluctuations and perform the ensemble averaging by assuming that ϕ is a Gaussian random variable and we get as the generalization of equation (2.43)

$$m_z^2\, \rho_I(\underline{\gamma}) + 1 = \left(\frac{k}{2\pi z}\right)^4 \iiiint d\underline{\xi}_1\, d\underline{\xi}_2\, d\underline{\xi}_3\, d\underline{\xi}_4 \;\times$$

$$F(\underline{\xi}_1,\underline{\xi}_2,\underline{\xi}_3,\underline{\xi}_4)\, V\left(\frac{\underline{\xi}_1-\underline{\xi}_2}{\lambda}\right) V\left(\frac{\underline{\xi}_3-\underline{\xi}_4}{\lambda}\right) \times$$

$$\exp\left[\frac{ik}{2z}\left\{\xi_1^2-\xi_2^2+(\underline{\xi}_3-\underline{\gamma})^2-(\underline{\xi}_4-\underline{\gamma})^2\right\}\right]$$

The function

$$G(\underline{\xi}_1,\underline{\xi}_2,\underline{\xi}_3,\underline{\xi}_4) = V\left(\frac{\underline{\xi}_1-\underline{\xi}_2}{\lambda}\right) V\left(\frac{\underline{\xi}_3-\underline{\xi}_4}{\lambda}\right) F(\underline{\xi}_1,\underline{\xi}_2,\underline{\xi}_3,\underline{\xi}_4)$$

is a function of only the difference of its arguments taken two at a time. So we can take the Fourier transform of the acf and using the result derived in appendix A we get

$$M_z^2(\underline{q}) + \delta(\underline{q}) = \int d\underline{\gamma}\; e^{i\underline{q}\cdot\underline{\gamma}}\; G\left(\underline{\gamma},\underline{\gamma}-\frac{z}{k}\underline{q},-\frac{z}{k}\underline{q},0\right)$$

where

$$G(\underline{r}, \underline{r} - \frac{z}{k}\underline{q}, -\frac{z}{k}\underline{q}, 0) = V(\frac{z}{k\lambda}\underline{q})\, V(-\frac{z}{k\lambda}\underline{q}) \times$$

$$F(\underline{r}, \underline{r} - \frac{z}{k}\underline{q}, -\frac{z}{k}\underline{q}, 0)$$

V is in general a complex function, but since its Fourier transform is the brightness distribution which is always real, V satisfied the property

$$V(-\underline{\omega}) = V^*(\underline{\omega})$$

So

$$V(\frac{z\underline{q}}{k\lambda})\, V(-\frac{z\underline{q}}{k\lambda}) = \left| V(\frac{z}{2\pi}\underline{q}) \right|^2$$

Since $\left| V(\frac{z}{2\pi}\underline{q}) \right|^2$ does not depend on $\underline{r}$ we can take it outside the integral and if we take the δ function to the right hand side within the integral we get

$$M_z^2(\underline{q}) = \left| V(\frac{z}{2\pi}\underline{q}) \right|^2 e^{-2\phi_0^2(1 - \rho(\frac{z}{k}\underline{q}))} \int d\underline{r}\, e^{i\underline{q}\cdot\underline{r}} \times$$

$$\left\{ \exp \phi_0^2 \left[2\rho(\underline{r}) - \rho(\underline{r} + \frac{z\underline{q}}{k}) - \rho(\underline{r} - \frac{z\underline{q}}{k}) \right] - 1 \right\}$$

$$= \left| V(\frac{z}{2\pi}\underline{q}) \right|^2 M_z(\underline{q})_{\text{point source}}$$

which is identical to equation (2.68).

From this expression we see that the effect of the finite source size is to filter out the higher spatial frequencies in the intensity pattern on the ground. Since the high spatial frequencies are attenuated, the second moment of the power spectrum is less than that for a point source i.e. the scale size on the ground is larger for a broad source than for a point source. Since the scintillation index is the area under the power spectrum, the scintillation index is smaller for an extended source.

So in principle the determination of the source structure is straight forward. All we have to do is measure the spatial power spectrum for a known point source and for the extended source. Dividing the second spectrum by the first we get the square of the visibility function of the source from which we can estimate the structure of the source. In practice however it is not possible to do this since the spatial power spectrum cannot easily be measured. What one usually has is some integral property of the spectrum like the scintillation index or the second moment and we have to estimate the structure of the source by comparing these with the corresponding quantities for a point source. For this we have to know how m_z and q_z change for different source models and this can be got only if we have an explicit expression for the point source power spectrum. As we have seen, we can get useful expressions for the spectrum only under limited conditions like $\phi_0 \ll 1$ or $z/z_0 \ll 1$. For arbitrary values of ϕ_0 and z/z_0 we can get estimates of the effect of the source size only by making ad hoc assumptions about the shape of the spectrum of the intensity fluctuations on the ground. Following Little and Hewish (1966) we will assume that the acf of the intensity fluctuations on the ground is a symmetrical Gaussian of the form $\rho_I(r) = \exp(-r^2/2\ell^2)$ and derive results for some simple source models. If a is the scale size on the screen we will take ℓ to be equal to a for $\phi_0 \ll 1$ and equal to a/ϕ_0 for $\phi_0 \gg 1$. The spatial power spectrum is proportional to $e^{-\ell^2 q^2/2}$

a) Circularly Symmetrical Gaussian Source

For a circularly symmetrical Gaussian source the brightness is given by

$$P(\theta_x, \theta_y) = \frac{1}{2\pi\theta_0^2} \exp - \frac{[\theta_x^2 + \theta_y^2]}{2\theta_0^2}$$

and

$$V(u, v) = \exp\left[- (2\pi\theta_0)^2 \frac{(u^2 + v^2)}{2} \right]$$

If m and l are the scintillation index and the scale size for a point source and m_s and l_s these quantities for an extended source using equation (2.68) we have

$$\frac{m_s^2}{m^2} = \frac{\int d\underline{q}\; e^{-l^2 q^2/2}\, e^{-\theta_0^2 z^2 q^2}}{\int d\underline{q}\; e^{-l^2 q^2/2}} = \frac{1}{1 + 2(z\theta_0/l)^2} \qquad (2.69a)$$

$$\frac{l_s^2}{l^2} = \frac{q_2^2}{q_{2s}^2} = \frac{\int d\underline{q}\; q_x^2\, e^{-\frac{l^2 q^2}{2}}}{\int d\underline{q}\; q_x^2\, e^{-\frac{l^2 q^2}{2}}\, e^{-\theta_0^2 z^2 q^2}} = 1 + 2(z\theta_0/l)^2 \qquad (2.69b)$$

These results show that the decrease in m and the increase in l is determined solely by the parameter $z\theta/l$. The scintillations are unaffected by the source size if $z\theta_0/l \ll 1$. If we take $l \approx a \approx 100$ km, z = 1 A.U. and define $\theta_{\frac{1}{2}}$ as the half power diameter of the source, the above condition becomes

$$\theta_{1/2} = 2.34\,\theta_0 \ll 2.34\; l/z \simeq 0.''15$$

If $z\theta_0/l \gg 1$ the scintillation index is reduced to zero and the scale size on the ground is determined purely by the diameter of the source and has a value $\sqrt{2}\, z\,\theta_0$. In the strong scattering regime l is equal to $a/\sqrt{2}\,\phi_0$ and so in this regime $\frac{z\theta_{\frac{1}{2}}}{\ell}$ increases linearly with ϕ_0. So even if we take a source with diameter $\theta_{\frac{1}{2}}$ much less than 0".15, when ϕ_0 is large enough to make $\phi_0\, \theta_{\frac{1}{2}} \gtrsim$ 0".15 the scintillation will start showing the effect of source size. If we take a source with diameter $\theta_{\frac{1}{2}} \ll$ 0".15 and study the variation of the scintillation index with ϕ_0, the scintillation index will increase with ϕ_0 in the weak scattering regime until it reaches a value of unity. As ϕ_0 increases m will remain constant at one until ϕ_0 is large enough to make $\phi_0\, \theta_{\frac{1}{2}} \sim$ 0".15 and then with increasi ϕ_0 the scintillation index will start reducing. For sources of different diameters the value of ϕ_0 at which m will start decreasing will be different. In the strong scattering regime the scale size decreases as ϕ_0^{-1} with increasing ϕ_0 until $\phi_0\, \theta_{\frac{1}{2}} \approx$ 0".15 and then it stays constant at a value equal to $\sqrt{2}\; z\,\theta$.

b) Elongated Gaussian Source

For an elongated Gaussian we have

$$p(\theta_x, \theta_y) = \frac{1}{2\pi\,\theta_{x_0}\theta_{y_0}} \exp -\frac{1}{2}\left[\frac{\theta_x^2}{\theta_{x_0}^2} - \frac{\theta_y^2}{\theta_{y_0}^2}\right]$$

If we carry out the calculation as in the previous section we get (Little and Hewish (1966))

$$\frac{m_s^2}{m_1^2} = \left[1 + 2\left(\frac{z\theta_{x0}}{\ell}\right)^2\right]^{-1/2}\left[1 + 2\left(\frac{z\theta_{y0}}{\ell}\right)^2\right]^{-1/2}$$

The scale size on the ground is no longer circularly symmetrical. For the x any y directions we have

$$\frac{\ell_{xs}^2}{\ell^2} = 1 + 2\left(\frac{z\theta_{x0}}{\ell}\right)^2 \;; \qquad \frac{\ell_{ys}^2}{\ell^2} = 1 + 2\left(\frac{z\theta_{y0}}{\ell}\right)^2$$

In this case also the scintillations behave as that of a point source provided both $\frac{z\theta_x}{\ell}$ and $\frac{z\theta_y}{\ell}$ are much less than one. When one of them approaches one m starts to decrease. The reduction in the scintillation index is much less rapid than in the circularly symmetrical case until both of them become greater than one and from this one can infer that the source is extended. The extended nature of the source and its dimensions can be estimated more directly by studying the shape of the irregularities on the ground because for an elongated source the irregularities on the ground are anisotropic.

c) Core halo sources

In core halo sources we have a core of intensity I_1, which may be a point or a symmetrical or an elongated Gaussian source, surrounded by a halo of intensity I_2, whose angular size is much greater than $\frac{a}{z} \simeq 0''.15$. The halo component does not scintillate and all the intensity fluctuations are due to the core. If we take the core to be a point source we can write the visibility function for the core halo source as

$$V(\underline{u}) = \frac{I_1}{I_1+I_2} + \frac{I_2}{I_1+I_2}\,\delta(\underline{u})$$

Using this we get

$$\frac{m_s^2}{m^2} = \frac{I_1^2}{(I_1+I_2)^2}\;, \qquad \frac{\ell_s}{\ell} = 1 \tag{2.70}$$

which shows that the scintillation index is reduced by a factor $I_1/(I_1+I_2)$ which is an obvious result since only this fraction of the incident flux contributes to the scintillation. The scale size is unaffected by the halo. The extension of the core halo model for cases where the core has a finite diameter is straight forward.

The interpretation of most of the sources observed in IPS will be made in terms of these three models. The use of more complicated models for the source is not usually justified since the IPS observations cannot, unambigously, decide between them.

2.10 Bandwidth Effect

In deriving equation (2.44) for the power spectrum it was assumed that the incident radiation was monochromatic. In practice the radiation is only quasi monochromatic i.e. there is a spread of frequencies but the bandwidth is generally much less than the central frequency. For quasi monochromatic radiation, the phase of the electric field at any point varies randomly with a time scale of $\tau_B \approx \frac{1}{\Delta\nu}$ where $\Delta\nu$ is the bandwidth. The temporal acf of the electric field $\rho_E(\tau) = \langle E(t)\, E^*(t+\tau)\rangle$ is the Fourier transform of the spectrum of the radiation which is the same as the bandpass of the receiver system. The effect of the finite bandwidth is to reduce the scintillation index because each frequency in the band produces a slightly different intensity pattern on the ground and the observed intensity pattern, which is the sum of the individual patterns, tends to get smoothed out.

An estimate of when the bandwidth will start affecting the intensity fluctuations can be got by the following arguments. The electric field at any point on the ground, say $r = 0$, is given by the sum of the wavelets from different parts of the phase screen. The wavelet coming from an element of area at ξ on the screen has to travel a distance $r = z + \xi^2/2z$ and so is delayed by a time $\xi^2/2zc$ with respect to the wavelet from $\xi = 0$. If this delay is larger than the coherence time $\tau_s = 1/\Delta\nu$, these two wavelets will not be coherent. As we have seen earlier, the largest value of ξ that affects us is $\xi_{max} \approx z\,\theta_{scat}$ and so for the decoherence due to the finite bandwidth to be negligible, we must have

$$(z\theta_{scat})^2/2zc \ll \tau_s$$

$$\text{i.e.} \quad \frac{z\,\theta_{scat}^2}{2c} \ll \frac{1}{\Delta\nu}$$

If ℓ is the scale size of the intensity fluctuations on the ground we can write $\theta_{scat} \sim \frac{\lambda}{2\pi\ell}$ and the above condition becomes $\frac{z\lambda^2}{8\pi^2\ell^2 c} \ll \frac{1}{\Delta\nu}$

which in practical units is

$$\frac{z_{A.U.}\,\lambda_m^2}{\ell_{100\,km}^2} \ll \frac{160\,\pi^2}{\Delta\nu_{MHz}} \tag{2.71}$$

For our observations if we take λ = 0.9 metre, z = 1 A.U., and $\Delta\nu$ = 4 MHz, which is the bandwidth used, the effect of the bandwidth is negligible provided

$$\ell \gg 4.5 \text{ km}$$

For elongations greater than 0.25 A.U. from the Sun the scale size, a, is around 100 km. and so the effect of bandwidth can be safely neglected in the weak scattering regime. In the strong scattering regime, since $\ell \approx a/\phi_0$, this is not always true and bandwidth starts playing an important role when ϕ_0 is greater than 20 or so.

The formal incorporation of the effect of bandwidth into the expression for the intensity is straightforward. If we denote by $E_0(\underline{\xi},t)$ the electric field at $\underline{\xi}$ on the screen at time t, then, by taking into account the differences in the time of arrival, equation (2.41) for the electric field on the ground can be written as

$$E(\underline{\xi},t) = \frac{k}{2\pi z}\, e^{ikz} \int d\underline{\xi}'\, E_0\left(\underline{\xi}', t - \frac{z}{c} - \frac{(\underline{\xi}'-\underline{\xi})^2}{2zc}\right) \times e^{i\frac{k}{2z}(\underline{\xi}'-\underline{\xi})^2}\, e^{i\phi(\underline{\xi}')}$$

The intensity is given by

$$I(\underline{\xi},t) = \left(\frac{k}{2\pi z}\right)^2 \iint d\underline{\xi}_1\, d\underline{\xi}_2\, E_0\left(\underline{\xi}_1, t - \frac{z}{c} - \frac{(\underline{\xi}_1-\underline{\xi})^2}{2zc}\right) E_0^*\left(\underline{\xi}_2, t - \frac{z}{c} - \frac{(\underline{\xi}_2-\underline{\xi})^2}{2zc}\right)$$
$$\times e^{i\frac{k}{2z}\left\{(\underline{\xi}_1-\underline{\xi})^2 - (\underline{\xi}_2-\underline{\xi})^2\right\}}\, e^{i(\phi(\underline{\xi}_1) - \phi(\underline{\xi}_2))}$$

Since any measurement of the intensity is always made with instruments whose time constant is much larger than $1/\Delta\nu$, using the same arguments as we did for the case of the finite source size, we will replace EE^* by its average value which is $\rho_t\left[\frac{(\underline{\xi}_1-\underline{\xi})^2-(\underline{\xi}_2-\underline{\xi})^2}{2zc}\right]$ where ρ_t is the temporal acf of the electric field and is the Fourier transform of the bandpass function. So we get

$$I(\underline{\xi}) = \left(\frac{k}{2\pi z}\right)^2 \iint d\underline{\xi}_1\, d\underline{\xi}_2\, \rho_t\left(\frac{(\underline{\xi}_1-\underline{\xi})^2 - (\underline{\xi}_2-\underline{\xi})^2}{2zc}\right)$$
$$e^{i(\phi(\underline{\xi}_1) - \phi(\underline{\xi}_2))}\quad e^{i\frac{k}{2z}\left\{(\underline{\xi}_1-\underline{\xi})^2 - (\underline{\xi}_2-\underline{\xi})^2\right\}}$$

The spatial acf is given by

$$m_{I_z}^2 \rho_I(\underline{\gamma}) + 1 = \left(\frac{k}{2\pi z}\right)^4 \iiiint d\underline{\xi}_1\, d\underline{\xi}_2\, d\underline{\xi}_3\, d\underline{\xi}_4\ F(\underline{\xi}_1, \underline{\xi}_2, \underline{\xi}_3, \underline{\xi}_4) \times$$

$$e^{i\frac{k}{2z}\left\{\underline{\xi}_1^2 - \underline{\xi}_2^2 + (\underline{\xi}_3 - \underline{\gamma})^2 - (\underline{\xi}_4 - \underline{\gamma})^2\right\}} \times$$

$$\rho_t\left\{\frac{(\underline{\xi}_1 - \underline{\xi})^2 - (\underline{\xi}_2 - \underline{\xi})^2}{2zc}\right\} \rho_t\left\{\frac{(\underline{\xi}_3 - \underline{\xi})^2 - (\underline{\xi}_4 - \underline{\xi})^2}{2zc}\right\}$$

ρ_t is a function of the $\xi_{i's}^2$ and is not a function just of the difference of the two arguments. So now we cannot, as we did earlier, make the various simplifications that were possible because the function $F(\underline{\xi}_i)$ depended only on the difference of its arguments. The best we can do is to take the Fourier transform of the above equation and get an expression for the power spectrum with 3 integrals in it, but further simplifications is not possible. Numerical integration of this triple integral is a formidable task. So though we have an expression for the effect of the bandwidth the extraction of useful information is not possible because of mathematical difficulties.

In the far field region where $z/z_0 \gg 1$ Little (1968) has derived some expressions for the effect of the bandwidth on the scintillation index and the shape of the acf by assuming that the angular spectrum produced by the screen is a Gaussian and that the phases of the different components in the spectrum are independent random variables, which is a good approximation for $z/z_0 \gg 1$. The results are expressed in terms of a parameter K which in our definition of a Gaussian is

$$K = \frac{2\pi\, \Delta\nu\, z\, \theta_{scat}^2}{c}$$

where $\Delta\nu$ is the width of the passband which is assumed to be a Gaussian. The factor b, by which the scintillation index is reduced, is given in both the weak and the strong scattering regimes as

$$b^2(K) = \sqrt{2\pi}/K \; \exp(1/2K)\left[1 - \Phi(1/K)\right]$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp(-t^2/2)\,dt$$

For $K \ll 1$ the scintillation index is unaffected and $b = 1$, but for $K \gg 1$ the expression for b becomes $b^2 \approx 1.25\,K^{-1}$.

The behaviour of the shape of the acf is different in the weak and strong scattering regimes. In the weak scattering regime the acf keeps on getting broader as K increases, showing that the effect of the bandwidth is to filter out the high frequencies in the spatial power spectrum. In the strong scattering regime the width does not increase as markedly as in weak scattering and for large K the width tends to saturate, becoming independent of the bandwidth. Thus in this regime the effect of the bandwidth is to attenuate all the spatial frequencies.

In the near field region no adequate theory exists which gives the variation of the observed quantities with bandwidth. But so long as the inequality (2.71) is satisfied, we can safely neglect the finite bandwidth and use the results got by assuming a monochromatic wave. All the above discussion is based on the thin screen model. For an extended medium we have no detailed theory. However when the total r.m.s. phase fluctuation ϕ_0 is much less than 1, we can replace the extended medium by a number of independent thin screens at different distances from the observer and the intensity pattern on the ground is simply the sum of the patterns produced by the individual screens. Under these conditions the bandwidth can be neglected if the inequality (2.71) is satisfied for all the screens. In the strong scattering regime multiple scattering between the screens has to be considered but even here, we can neglect the bandwidth if we take the most distant screen and put $\theta_{scat} = \dfrac{\phi_0 \lambda}{2\pi a}$ and find that the inequality (2.71) is satisfied.

2.11 Moving Screen

So far we have not bothered about how the intensity pattern on the ground was actually measured. If the screen is stationary, one must either have a large number of detectors at different places on the ground or one must have a single detector to various points on the ground in order to measure the intensity pattern. Both these alternatives are quite impractical and useful observations of IPS are possible only because the solar plasma is flowing away from the Sun in a fairly well ordered way, causing the intensity pattern on the ground to also drift with the same velocity. So the intensity pattern on the ground varies with time and this can be observed easily with a single antenna. If we assume that the time variation in the intensity pattern is due only to the motion of the screen, we can infer the spatial dependence of the intensity from the observed time variations. For this one has to know the velocity of the solar wind which has been estimated by observing the intensity fluctuations at 3 well separated points on the ground and by measuring the difference in the time at which the pattern reaches the 3 stations. However the idealised situation where the intensity pattern drifts without changing is unlikely to be a good approximation and so we will first consider the general case where the pattern changes while drifting and see how the temporal acf of the intensity fluctuations is related to the spatial acf.

We will again consider the thin screen model where $\phi(\underline{\xi},\tau)$ is the variation of the phase about the mean value at the point $\underline{\xi}$ at time t. The general acf is given by

$$\phi_0^2 \, \rho(\underline{\gamma},\tau) = < \phi(\underline{\xi},t) \, \phi(\underline{\xi}+\underline{\gamma},t+\tau) >$$

The intensity at any point on the ground at time t is (equation (2.42))

$$I(\underline{\xi},t) = \left(\frac{k}{2\pi z}\right)^2 \iint d\underline{\xi}_1 \, d\underline{\xi}_2 \; e^{i\frac{k}{2z}\left\{(\underline{\xi}_1-\underline{\xi})^2-(\underline{\xi}_2-\underline{\xi})^2\right\}} \; e^{i\left\{\phi(\underline{\xi}_1,t)-\phi(\underline{\xi}_2,t)\right\}}$$

The generalised acf of the intensity fluctuation on the ground is

$$m_z^2 \, \rho_I(\underline{\gamma},\tau) = < I(\underline{\xi},t) \, I(\underline{\xi}+\underline{\gamma},t+\tau) > - 1$$

In all our earlier discussions we have been dealing with $\rho_I(\underline{\gamma},\tau=0)$ while what we measure with a single antenna is $\rho_I(\underline{\gamma}=0,\tau)$ and we have to see how the two are related. If we regard $\phi(\underline{\xi},\tau)$ to be a random Gaussian variable, we can generalise equation (2.43) and get

$$m_z^2 \, \rho_I(\underline{\gamma},\tau) + 1 = \left(\frac{k}{2\pi z}\right)^4 \iiiint d\underline{\xi}_1 \, d\underline{\xi}_2 \, d\underline{\xi}_3 \, d\underline{\xi}_4$$

$$\exp \frac{ik}{2z}\left\{\underline{\xi}_1^2-\underline{\xi}_2^2+(\underline{\xi}_3-\underline{\gamma})^2-(\underline{\xi}_4-\underline{\gamma})^2\right\} \; F(\underline{\xi}_1,\underline{\xi}_2,\underline{\xi}_3,\underline{\xi}_4,\tau)$$

where

$$F(\underline{\xi}_1,\underline{\xi}_2,\underline{\xi}_3,\underline{\xi}_4,\tau) = \exp -\phi_0^2 \left\{2-\rho(\underline{\xi}_1-\underline{\xi}_2,0) + \rho(\underline{\xi}_1-\underline{\xi}_3,\tau) - \rho(\underline{\xi}_1-\underline{\xi}_4,\tau) - \rho(\underline{\xi}_2-\underline{\xi}_3,\tau) + \rho(\underline{\xi}_4-\underline{\xi}_2,\tau) - \rho(\underline{\xi}_3-\underline{\xi}_4,0)\right\}$$

Let us define

$$N(\underline{q},\tau) = \int d\underline{\gamma} \; e^{i\underline{q}\cdot\underline{\gamma}} \; m_z^2 \, \rho(\underline{\gamma},\tau)$$

Taking the Fourier transform of the acf of the intensity fluctuations and noting that $F(\xi_1, \xi_2, \xi_3, \xi_4, \tau)$ is a function of only the difference of the ξ_i's so that all the simplifications we made earlier can still be made, we get

$$N(\underline{q},\tau) = \int d\underline{r}\; e^{i\underline{q}\cdot\underline{r}} \left\{ F(\underline{r}, \underline{r} - \tfrac{z}{k}\underline{q}, -\tfrac{z}{k}\underline{q}, 0, \tau) - 1 \right\}$$

$$= \exp - \left[2\phi_0^2 \left(1 - \rho(\tfrac{z}{k}\underline{q}, 0)\right)\right] \int d\underline{r}\; e^{i\underline{q}\cdot\underline{r}} \times$$

$$\left[\exp \phi_0^2 \left\{ 2\rho(\underline{r},\tau) - \rho(\underline{r} + \tfrac{z}{k}\underline{q}, \tau) - \rho(\underline{r} - \tfrac{z}{k}\underline{q}, \tau) \right\} - 1 \right] \qquad (2.72)$$

The acf of the intensity fluctuations is given by

$$m_z^2 \; \rho_I(\underline{r},\tau) = \int d\underline{q}\; N(\underline{q},\tau)\; e^{-i\underline{q}\cdot\underline{r}} \qquad (2.73)$$

The temporal acf is given by

$$m_z^2 \; \rho_I(0,\tau) = \int d\underline{q}\; N(\underline{q},\tau)$$

The scintillation index estimated from the time variations of the intensity at a point is given by

$$m_z^2 = m_z^2 \; \rho_I(0,0) = \int d\underline{q}\; N(\underline{q},0)$$

which is identical to the scintillation index as estimated from the spatial variations of the intensity at a given instant of time.

The situation is more complicated in the case of the actual acf and the power spectra. In view of the complexity of $N(\underline{q},\tau)$ we will consider only two limiting cases 1) $\phi_0 << 1$ 2) $z/z_0 >> 1$

1) $\Phi_0 << 1$

Under this approximation $N(\underline{q},\tau)$ becomes

$$N(\underline{q},\tau) = 4\, \Phi_s^2(\underline{q},\tau)\, \sin^2\left(\frac{zq^2}{2k}\right) \qquad (2.74)$$

where

$$\Phi_s^2(\underline{q},\tau) = \int d\underline{r}\; e^{i\underline{q}\cdot\underline{r}}\; \phi_0^2\, \rho(\underline{r},\tau) \tag{2.75}$$

the temporal acf is given by

$$m_z^2\rho_I(0,\tau) = \int d\underline{q}\, N(\underline{q},\tau) = 4\int d\underline{q}\, \overline{\Phi}_s^2(\underline{q},\tau)\sin^2\left(\frac{zq^2}{2k}\right) \tag{2.76}$$

while the temporal power spectrum is given by

$$P(\omega) = \int e^{i\omega\tau} m_z^2 \rho_I(0,\tau)\,d\tau = 4\int d\underline{q}\,\overline{\Phi}^2(\underline{q},\omega)\sin^2\left(\frac{zq^2}{2k}\right) \tag{2.77}$$

where $\overline{\Phi}^2(\underline{q},\omega)$ is the space time Fourier transform of the acf of the phase fluctuations, defined by

$$\overline{\Phi}^2(\underline{q},\omega) = \phi_0^2 \iint d\underline{q}\,d\tau\; e^{i(\underline{q}\cdot\underline{r}+\omega\tau)}\rho(\underline{r},\tau) \tag{2.78}$$

2) $z/z_0 \gg 1$

Under this approximation we can go through the same arguments as was done from the Far field approximation in the time independent screen and we get

$$N(\underline{q},\tau) = e^{-2\phi_0^2}\int d\underline{r}\; e^{i\underline{q}\cdot\underline{r}}\left[\exp\{2\phi_0^2\rho(\underline{r},\tau)\} - 1\right]$$

which gives

$$m_z^2\rho_I(\underline{r},\tau) = e^{-2\phi_0^2}\left[e^{2\phi_0^2\rho(\underline{r},\tau)} - 1\right] \tag{2.79}$$

So far the discussion has been quite general but now we have to make some model for the time varying pattern and assume some form for $\rho(\underline{r}, \tau)$, before we can relate the quantities measured with a single antenna with the parameters of the screen. We will consider in some detail the model in which the pattern is assumed to drift without changing and then discuss briefly the more general forms of the acf.

2.11a <u>Moving Screen with Time Independent Phase Pattern</u>

In this model we assume that the phase fluctuations on the screen are independent of time and consider the screen as a whole to move with a uniform velocity $\underline{V}$. In the frame of reference in which the screen is at rest, the acf of the phase fluctuations is independent of time and will be denoted by $\rho(\underline{r})$. In the frame of reference in which the screen is moving, the acf is a function of both space and time, but it has the simple form

$$\rho(\underline{r}, \tau) = \rho(\underline{r} - \underline{V}\tau)$$

The space time Fourier transform of the acf has the form

$$\Phi^2(\underline{q}, \omega) = \iint d\underline{r}\, d\tau\, e^{i(\underline{q}\cdot\underline{r} + \omega\tau)} \rho(\underline{r} - \underline{V}\tau) = \Phi^2(\underline{q})\, \delta(\omega + \underline{q}\cdot\underline{V})$$

If we insert this into equation (2.77) for the power spectrum and if we choose the coordinate system so that $\underline{V}$ is in the direction of the x axis, we find that for $\phi_0 \ll 1$

$$P(\omega) = 4 \int d\underline{q}\, \Phi^2(\underline{q})\, \delta(\omega + V q_x)\, \sin^2\left(\frac{z q^2}{2k}\right)$$

$$= \frac{4}{V} \int dq_y\, \Phi\left(\frac{\omega}{V}, q_y\right) \sin^2\left\{\frac{z}{2k}\left(\frac{\omega^2}{V^2} + q_y^2\right)\right\}$$

In practice the temporal power spectrum is generally expressed in terms of frequency f rather than angular frequency ω. Since $\omega = 2\pi f$ we have

$$P(f) = 2\pi \; P\left(f = \frac{\omega}{2\pi}\right)$$

$$= \frac{8\pi}{V} \int_{-\infty}^{\infty} dq_y \; \Phi^2\left(\frac{2\pi f}{V}, q_y\right) \sin^2\left\{\frac{z}{2k}\left[\left(\frac{2\pi f}{V}\right)^2 + q_y^2\right]\right\}$$

From the above expression we see that the temporal power spectrum is related to the 2 dimensional power spectrum in the same way as a strip scan over a radio source is related to the actual 2 dimensional structure of the source. A consequence of this is that the Fresnel modulation is smeared out and so is not very prominent in the observed temporal power spectrum.

It is of interest, however, to see the Fresnel modulation in the power spectrum because if one sees the nulls of the $\sin^2$ term one can estimate the velocity of the screen by assuming only the distance to the screen. If we assume that the 2 dimensional spatial power spectrum is circularly symmetrical we know, from the theory of the restoration of strip scans (Bracewell, 1965) that the Abel transform of the observed strip scan gives the circularly symmetrical 2 dimensional structure of the source. So if we take the Abel transform of the observed temporal power spectrum we get

$$G(f) = \text{Abel Transforms of } P(f) = -\frac{1}{\pi} \int_{f}^{\infty} df' \, \frac{dP(f')}{df'} \Big/ \sqrt{f'^2 - f^2} \qquad (2.80)$$

$$= k \, \Phi^2\left(\frac{2\pi f}{V}\right) \sin^2\left\{\frac{z}{2R}\left(\frac{2\pi f}{V}\right)^2\right\}$$

Thus the Fresnel modulation is more clearly seen on the Abel transform. Taking the Abel transform of the Fourier transform of a function is equivalent to the single operation of taking the Bessel transform of the function. So the above result could also have been got by taking the Bessel transform of the observed temporal acf i.e.

$$G(f) = \int_0^\infty d\tau \, \tau \, J_0(2\pi f \tau) \, \rho_I(0,\tau) \qquad (2.81)$$

where $J_0(x)$ is the Bessel function of zeroth order.

For the case of a Gaussian acf, in the weak scattering limit, we can derive an explicit expression for the temporal power spectrum. Let the acf of the phase fluctuations be of the form

$$\rho(x,y) = \exp - (x^2+y^2)/2a^2$$

The power spectrum of the phase fluctuations is

$$\Phi^2(q_x,q_y) = \frac{a^2\phi_0^2}{2\pi} \exp - \frac{a^2}{2}(q_x^2+q_y^2)$$

The spatial power spectrum of the intensity fluctuations on the ground is, in the weak scattering limit, given by

$$M_z^2(q_x,q_y) = \frac{2\phi_0^2}{\pi} \exp\left[-\frac{a^2}{2}(q_x^2+q_y^2)\right] \sin^2\left\{\frac{z\lambda}{4\pi}(q_x^2+q_y^2)\right\}$$

The temporal power spectrum is given by

$$P(f) = \frac{4a^2\phi_0^2}{V} \int_{-\infty}^{\infty} dq_y \, e^{-\frac{a^2}{2}(q_x^2+q_y^2)} \sin^2\left\{\frac{z\lambda}{4\pi}(q_x^2+q_y^2)\right\} \; ; \; q_x = \frac{2\pi f}{V}$$

The integral over q_y can be performed in a fairly straight forward way by expressing the sin term as the sum of two exponentials, collecting the terms containing q_y^2, expressing them as a perfect square and integrating over the resulting Gaussian functions. The resulting expression is

$$P(f) = \sqrt{8\pi}\ \phi_0^2\ \frac{a}{V}\ e^{-\frac{a^2}{2}\left(\frac{2\pi f}{V}\right)^2} \times$$

$$\left[1 - \frac{\cos\left\{\frac{z\lambda}{2\pi}\left(\frac{2\pi f}{V}\right)^2 + \psi/2\right\}}{\left(1+\left(2z/z_0\right)^2\right)^{1/4}}\right] \qquad (2.82)$$

where $z_0 = ka^2$ and $\psi = \tan^{-1}(2z/z_0)$

The power spectrum as we have defined it has both positive and negative frequencies, i.e. f goes from $-\infty$ to $+\infty$. It is customary to define the power spectrum for only positive frequencies, in which case, since the area under the power spectrum should be equal to the variance, the equation (2.82) for the power spectrum should be multiplied by two.

The second moment of the temporal power spectrum

$$f_2^2 = \int_0^\infty df\, f^2\, P(f) \Big/ \int_0^\infty df\, P(f)$$

can be got by inserting the above expression for P(f) and by performing the integration. However, if we substitute for P(f) from equation (2.79a) we see that the expression for f_2 is the same as that in equation (2.48) with $2\pi f/V$ in the place of q_x, so that from equation (2.56) we have

$$f_2 = \frac{V}{2\pi a}\left[1 + 2/1 + \left(2z/z_0\right)^2\right]^{1/2} \qquad (2.83)$$

The extension of these results to an unsymmetrical Gaussian phase acf is straightforward. If we assume that

$$\rho(x,y) = \exp - \left\{\frac{x^2}{2a^2} + \frac{y^2}{2b^2}\right\}$$

we can show that

$$P(f) = \sqrt{8\pi}\ \phi_0^2\ \frac{a}{V}\ e^{-\frac{a^2}{2}\left(\frac{2\pi f}{V}\right)^2} \times$$

$$\left[1 - \frac{\cos\left\{z\lambda/2\pi\left(2\pi f/V\right)^2 + \psi/2\right\}}{\left(1+\beta^2\right)^{1/4}}\right] \qquad (2.84)$$

where $\Psi = \tan^{-1}\beta$ and $\beta = 2z/kb^2$

From this expression we see that while the width of the power spectrum is determined mainly by the scale size in the direction of motion of the screen, the amplitude of the Fresnel modulation is determined by the scale size in the perpendicular direction. When $b \to 0$ i.e. $\beta \to \infty$ the Fresnel modulation vanishes. When $b \to \infty$ or $\beta \to 0$, the two dimensional screen reduces to a one dimensional screen and the power spectrum takes the form

$$P(f) = \sqrt{32\pi}\, \phi_0^2 \frac{a}{V} e^{-\frac{a^2}{2}\left(\frac{2\pi f}{V}\right)^2} \sin^2\left\{\frac{z}{2k}\left(\frac{2\pi f}{V}\right)^2\right\}$$

The scintillation index and the second moment of the temporal power spectrum for the unsymmetrical Gaussian acf are given by

$$m^2 = 2\phi_0^2\left[1 - \cos\Psi/2 \times \sqrt{\cos\theta_x \cos\theta_y}\right]$$

$$f_2^2 = \left(\frac{V}{2\pi a}\right)^2 \left\{\frac{1 - \cos(\theta_x + \Psi/2) \times \sqrt{\cos^3\theta_x \cos\theta_y}}{1 - \cos\Psi/2 \times \sqrt{\cos\theta_x \cos\theta_y}}\right\}$$

where we have substituted $\tan\theta_x = 2z/ka^2$, $\tan\theta_y = 2z/kb^2$ and $\Psi = \theta_x + \theta_y$

It is of interest to see how the Bessel transform will look when the irregularities are not circularly symmetrical. Bourgois (1972) has derived an expression for the Bessel transform, which in our notation is

$$G(f) = \left(\frac{\phi_0}{\pi}\right)^2 \frac{1}{\alpha} e^{-\frac{\alpha^2}{2}\left(\frac{2\pi f}{V}\right)^2}\left\{1 - \sqrt{\frac{\cos\theta_y}{\cos\theta_x}} \cos\left[\frac{z}{k}\left(\frac{2\pi f}{V}\right)^2 + \frac{\theta_y - \theta_x}{2}\right]\right\} \tag{2.85}$$

When $\alpha = \beta$ we get back the circularly symmetrical case. When $\alpha < \beta$ the amplitude of the modulation is reduced. When $\alpha > \beta$ the amplitude increases and the Bessel transform even goes negative for certain values of f. In principle, by studying the amplitude of the Fresnel modulation, one could deduce the elongation of the irregularities, but, as we will see later, there are other processes that reduce the amplitude of the modulation and it is not possible, from single station observations to unravel these effects.

2.11b Effects of an Evolving Pattern

The assumption that the phase pattern on the screen moves without changing is unlikely to be correct for the actual IP medium. In the Geometrical optics approximation the phase pattern on the screen is related to the density fluctuations in the medium by

$$\phi(x,y,t) = \frac{2\pi}{\lambda} \int_{-L/2}^{L/2} n(x,y,z,t)\, dz \qquad (2.86)$$

Two conditions have to be satisfied for the phase pattern to drift without changing its shape. The first is that the density fluctuations themselves should drift without changing their shape, which is unlikely to be true in the IP medium since because of diffusion and other processes the density fluctuations would get smoothed out. However, if the time scale for this smoothing is more than a few seconds, we can, without much error, assume that the density fluctuations are frozen into the medium and write

$$n(x,y,z,t) = n(x-V\,t,y,\,z)$$

where V, the velocity of the medium perpendicular to the line of sight, is assumed to be in the x direction. Even if the density fluctuations are